Two-phase flow in porous media: dynamic capillarity and hysteresis

Xiulei Cao
Supervisor: I.S. Pop.
Centre for Analysis, Scientific computing and Applications (CASA)

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Outline

Motivation

Mathematical model

Results in homogeneous media
  Dynamic effects and constant total flow
  Dynamic effects and degenerate case
  Dynamic effects and hysteresis

Heterogeneous media

Future work
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Figure 1: The setup (a) and saturation overshoot (c).

Fingering and Overshoot

Figure 2: The setup (a) flow path (b) and saturation overshoot (c).

*F. Rezanezhad, H.-J. Vogel and K. Roth, Experimental study of fingered flow through initially dry sand.

Non-equilibrium $p - s$ curve

Figure 3: The water saturation (left) and phase pressure difference (right).

**Standard:**  
\[ p = p_c(s) \text{ (monotonic).} \]

**Non-standard:**  
\[ p = p_c(s, \partial_t s, ...) \]

*S. Bottero, Advances in the Theory of Capillarity in Porous Media.*
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Mathematical model (two phases, water/oil)

Darcy

\[ \mathbf{v}_\alpha = -\frac{k}{\mu_\alpha} k_{r\alpha} (\nabla p_\alpha + \mathbf{g}), \quad \alpha = w, \ n. \]

Mass balance

\[ \frac{\partial}{\partial t} (\phi s_\alpha) + \nabla \cdot \mathbf{v}_\alpha = 0, \quad \alpha = w, \ n. \]

Gives

\[ \frac{\partial}{\partial t} (\phi s_\alpha) - \nabla \cdot \left( \frac{k}{\mu_\alpha} k_{r\alpha} (\nabla p_\alpha + \mathbf{g}) \right) = 0, \quad \alpha = w, \ n. \]

Note:

\[ s_w + s_n = 1. \]
Non-equilibrium effects

Standard model:

\[ p_n - p_w = p_c(s_w). \]

Non-equilibrium model:

\[ p_n - p_w = p_c(s_w) - \tau \frac{\partial s_w}{\partial t} - \gamma \text{sign}\left(\frac{\partial s_w}{\partial t}\right), \quad \tau > 0, \gamma \geq 0. \]

Dimensionless model

Reference values: $L, Q, T, P, K$ with $Q = \frac{L\phi}{T}$, dimensionless quantities $x := \frac{x}{L}, t := \frac{t}{T}$

\[
\begin{align*}
\partial_t s - \nabla \cdot \left( k_w(s)(\nabla p_w + g) \right) &= 0, \\
-\partial_t s - \nabla \cdot \left( k_n(s)(\nabla p_n + g) \right) &= 0, \\
p_n - p_w &= p_c(s) - \tau \partial_t s - \gamma \text{sign}(\partial_t s),
\end{align*}
\]

with

\[s := s_w, \quad k_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}, \quad \alpha = w, n.\]

Note:

\[
\nabla \cdot \vec{q} = 0,
\]

with $\vec{q} = k_w(s)(\nabla p_w + g) + k_n(s)(\nabla p_n + g)$. 

\[\]
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Simplified model: $\gamma = 0$ and total flow constant

$$
\partial_t s + \nabla \cdot F(s) + \nabla \cdot \left( H(s) \nabla \left( p_c(s) - \tau \partial_t s \right) \right) = 0, \text{ in } (0, T] \times \Omega,
$$
e.g.: two-phase flow model, constant total flow $\vec{q}$

$$
F(s) = \vec{q} \frac{k_w}{k}(s), \quad H(s) = \frac{k_w k_n}{k}(s), \quad \text{and} \quad k = k_w + k_n.
$$

**Theorem 1**

*With sufficient regular boundary and initial conditions, the non-degenerate model $(0 < m \leq H \leq M < \infty)$ has at most one weak solution $s \in W^{1,2}(0, T; W^{1,2}(\Omega))$.  

*Note:*

$$
s|_{\partial \Omega} = s_D \in C^{1,\alpha}(\bar{\Omega}), \quad s(0, \cdot) = s_0 \in C^{0,\alpha}(\bar{\Omega}).
$$

*A. Mikelić: A global existence result for the equations describing unsaturated flow in porous media with dynamic capillary pressure, J. Differ. Equ. 248, 1561-1577(2010)*
Degenerate case ($\gamma = 0, \tau = \tau(s)$):

\[ k_w(s) \sim s^\alpha, \quad k_n(s) \sim (1 - s)^\beta, \quad -p'_c \sim s^{-\lambda}, \quad \tau(s) \sim (1 - s)^{-\omega}. \]

"Old" form uses

\[ p_g = p_n - \int_{C_D}^s f_w(z)p'_c(z)dz, \quad \theta(s) = -\int_{C_D}^s \frac{k_wk_n}{k}(z)p'_c(z)dz, \]

to transform the model into

\[ \partial_t s + \nabla \cdot \left( k_n(s)\nabla p_g \right) - \Delta \theta(s) = 0, \]

\[ \nabla \cdot \left( k(s)\nabla p_g \right) + \nabla \cdot \left( k_w(s)\nabla \left( \tau(s)\partial_t s \right) \right) = 0. \]
Existence of weak solutions

Theorem 2

With proper assumptions for the coefficients, \( s|_{\partial \Omega} = C_D \in (0, 1) \) and \( s(0, \cdot) = s_0 \in (0, 1) \), there exists a weak solution pair \((s, p_g)\) for the model which satisfies:

\[
\begin{align*}
    s &\in L^2(0, T; W^{1,2}_{C_D}(\Omega)), \quad \partial_t s \in L^2(\Omega \times (0, T)), \\
p_g &\in L^2(0, T; W^{1,r^*}_0(\Omega)) \text{ (for some } r^* \in (1, 2)).
\end{align*}
\]

Note: given \( p_g \), one has

\[
p_n = p_g + \int_{C_D}^s f_w(z)p'_c(z)dz, \quad p_w = p_n - p_c(s) + \tau \frac{\partial s}{\partial t}.
\]
\( \gamma := \gamma(x) \geq 0, \tau > 0 \)

\[
\begin{align*}
\partial_t s - \nabla \cdot \left( k_w(s)(\nabla p_w + g) \right) &= 0, \\
-\partial_t s - \nabla \cdot \left( k_n(s)(\nabla p_n + g) \right) &= 0, \\
p_n - p_w &= p_c(s) - \tau \partial_t s - \gamma(x) \text{sign}(\partial_t s).
\end{align*}
\]

**Theorem 3**

*With sufficient regular boundary and initial conditions, under non-degenerate case there exists at most one weak solution \((s, p_w, p_n)\) for the model.*

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2 rocks - heterogeneous media:

Figure 4: Heterogeneous media

- **Problem:** Discontinuous coefficients appear under 2\textsuperscript{nd} order derivatives.
- **Approach:** Models/sub-domains.
- **Q:** How should the models be coupled at interface?

* C.J. van Duijn, Molenaar and M.J. de Neef, The effect of capillary forces on immiscible two-phase flow in heterogeneous media.

1-D heterogeneous model $\gamma = 0, \tau > 0$:

\[
\begin{aligned}
\phi \frac{\partial s}{\partial t} + \frac{\partial F}{\partial x} &= 0, \\
F &= qf_w(s) + \bar{k}(x)\bar{\lambda}(s) \frac{\partial}{\partial x} \left( \frac{J(s)}{h(x)} - \tau \frac{\partial s}{\partial t} \right),
\end{aligned}
\]

with $h(x) = \sqrt{\frac{\bar{k}(x)}{\phi}}$. 
Assumptions

\( J, h \)

**Figure 5:** The functions: \( J \) (a) and \( h \) (b).

\textbf{Id:} approximate the interface \((x = 0)\) by a thin layer.
Deriving the coupling conditions

- Regularization (blow up the interface $x = 0 \rightarrow (-\epsilon, \epsilon)$):

  \[ h_\epsilon(x) = \hat{h}(\frac{x}{\epsilon}) \quad \text{for} \quad -\epsilon < x < \epsilon. \]

  Here $h_\epsilon$ is like

![Diagram showing the behavior of $h_\epsilon$](image)

**Figure 6:** $h_\epsilon$ function

Find limit $\epsilon \downarrow 0$. 
Coupling conditions:

- **A1:** Flux continuity.
- **A2:** Pressure condition (entry pressure):
  - pressure continuity, or no oil flowing into the fine medium.

\[ \tau = 0 : \]

\[ (p^- - p^+) (1 - s^+) = 0. \]

However, oil may still flow even if \( s > s^* \) (less oil trapped).

**Figure 7:** Capillary pressure with entry pressure (Standard model: \( p_n - p_w = p_c(s) \)).
Numerical results

$t = 0.7$:

![Graph showing oil saturation over time](a)

![Graph showing oil flow over time](b)

**Figure 8**: Oil trapped for $\tau = 0$ (a) and oil flowing for $\tau = 1$ (b).
$t = 400:$

**Figure 9:** Oil saturation: $\tau = 0$ (a), $\tau = 10$ (b) and $\tau = 30$ (c).
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Future work
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- Finite volume analysis?
- Finite element analysis?
- Writing thesis!

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Thank you for your attention!