Level Set Methods and Fast Marching Methods

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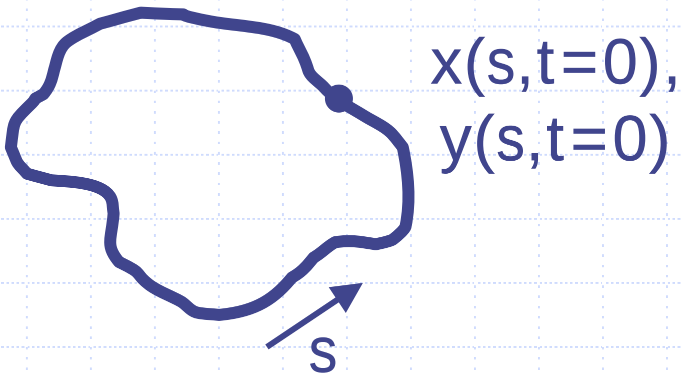
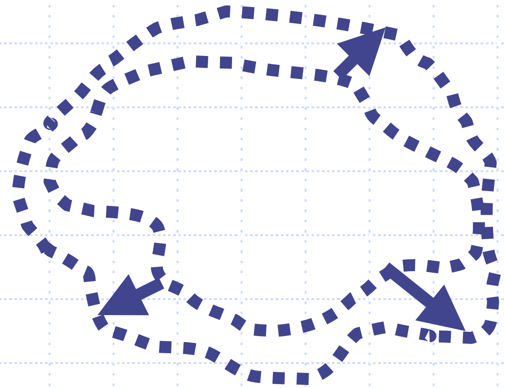
Overview

- Existing Techniques for Tracking Interfaces
- Basic Ideas of Level Set Method and Fast Marching Method
- Linking moving fronts and hyperbolic conservation laws

Tracking a moving boundary
Lagrangian approach

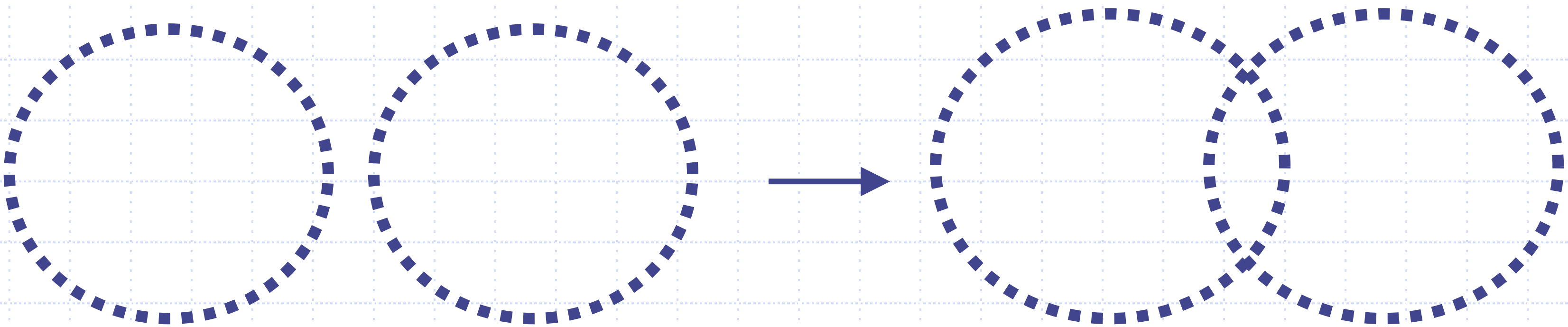


parameterization of the curve:
(x(s,t),y(s,t))



discrete
parameterization of the curve

?? How to deal with topological changes?

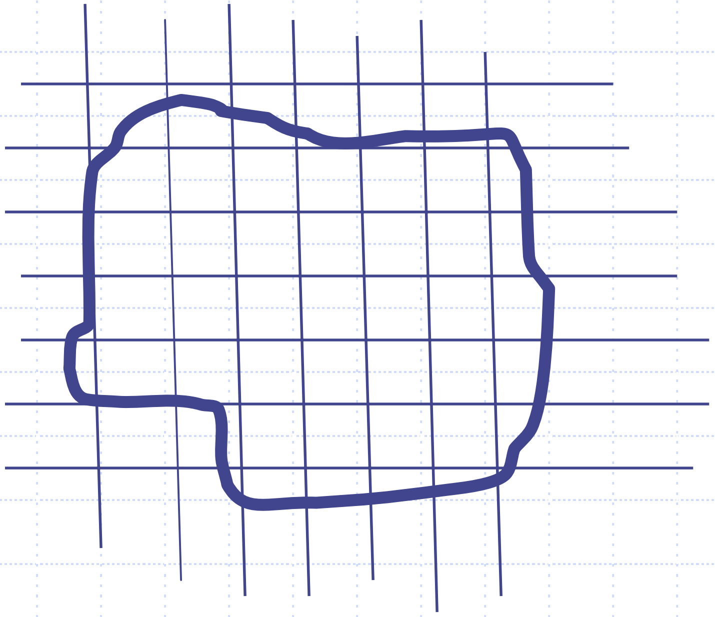


Tracking a moving boundary

Eulerian approach

Volume-of-fluid method:

.5	.9	.7	.3	.2	.3	.2
1	1	1	1	1	1	.6
1	1	1	1	1	1	.7
1	1	1	1	1	1	.8
.9	.9	1	1	1	1	.8
0	.2	1	1	1	1	.7
0	.1	1	1	1	1	.5



?? Drawbacks:

-- approximation to the front is crude, a large number of cells

-- curvature and normal is difficult to derive

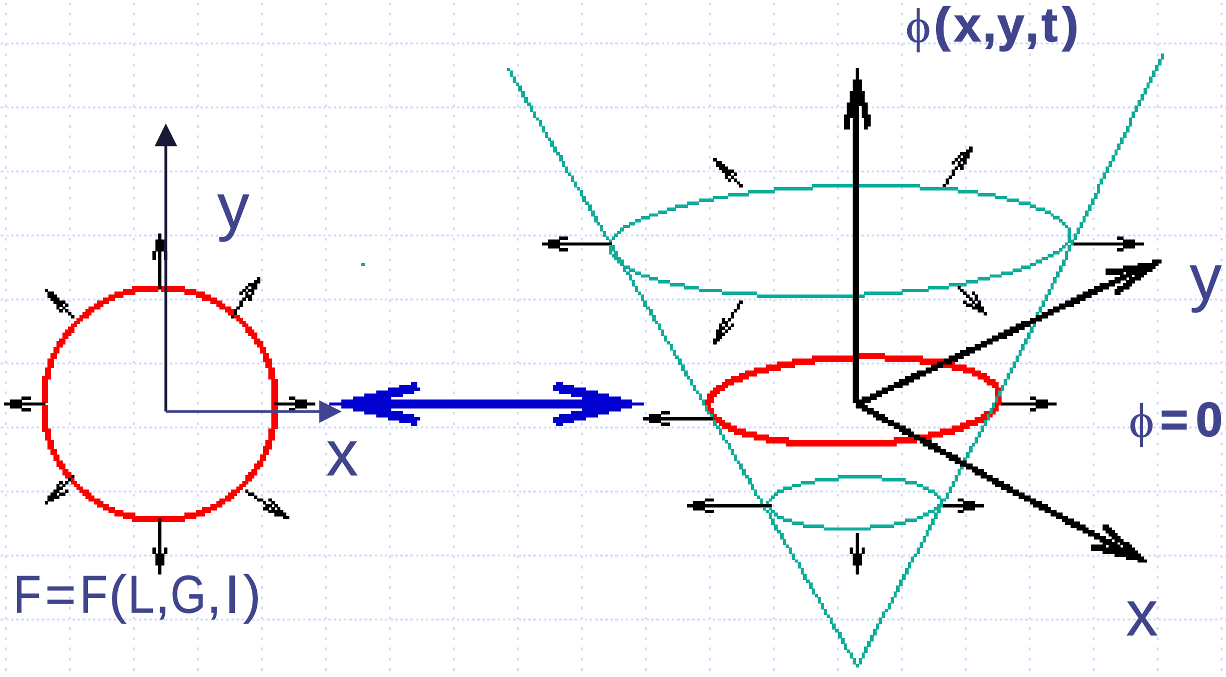
-- in 3D very complicated to perform

Level set and Fast marching methods

Sethian J. A. Level Set Methods and Fast Marching Methods: Evolving Interfaces in Geometry, Fluid Mechanics, Computer Vision and Material Science. *Cambridge University Press, 1999*

http://math.berkeley.edu/~sethian/level_set.html

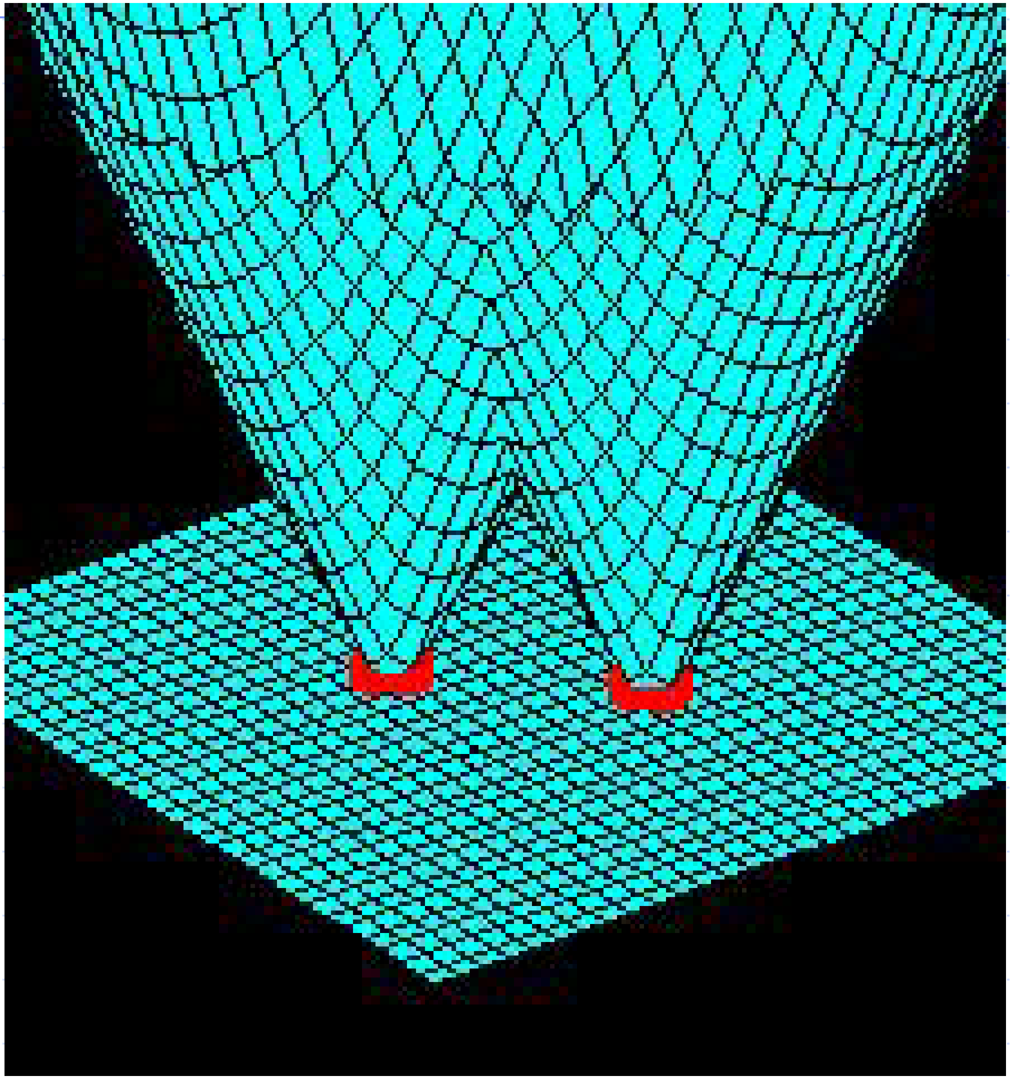
Level Set Method:
an initial value formulation



original front

level set function

How do you move the front?



Why is this called an "initial value formulation"?

Level set equation:

$$x(t) : \phi(x(t),t)=0 \qquad \phi_t + \frac{\partial \phi}{\partial x} \cdot x'(t) = 0$$

If front moves in normal direction:

$$\bar{n} = \frac{\nabla \phi}{|\nabla \phi|} \qquad F = \bar{n} \cdot x'(t)$$

$$\phi_t + F\,|\nabla \phi| = 0 \qquad IC : \phi(x, t = 0)$$

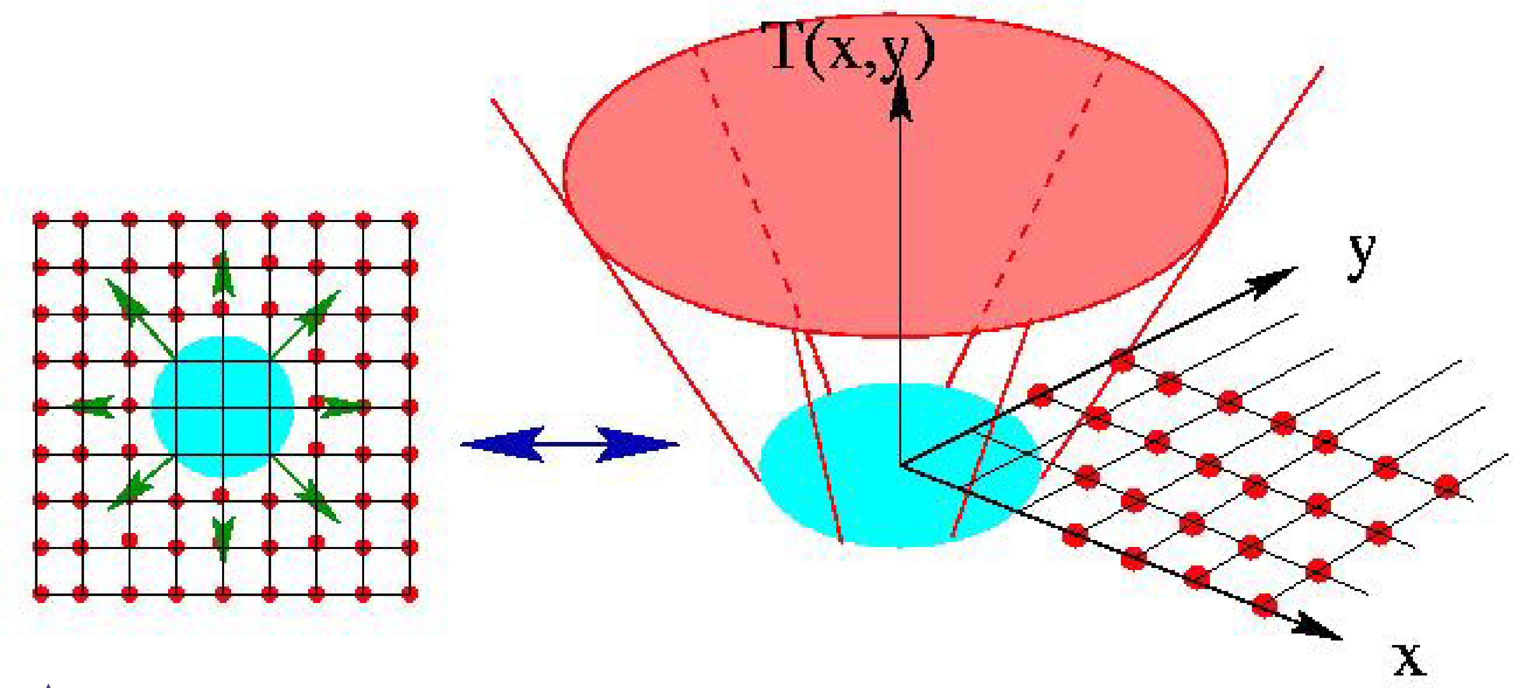
If front is advected by velocity field:

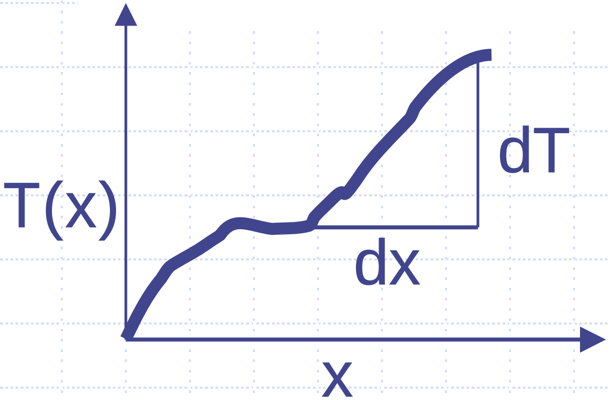
$$\bar{F} = (u, v)$$

$$\phi_t + \bar{F} \cdot \nabla \phi = 0 \qquad \phi_t + u \cdot \phi_x + v \cdot \phi_y = 0$$

$$IC : \phi(x, t = 0)$$

Fast Marching Method:
a boundary value formulation

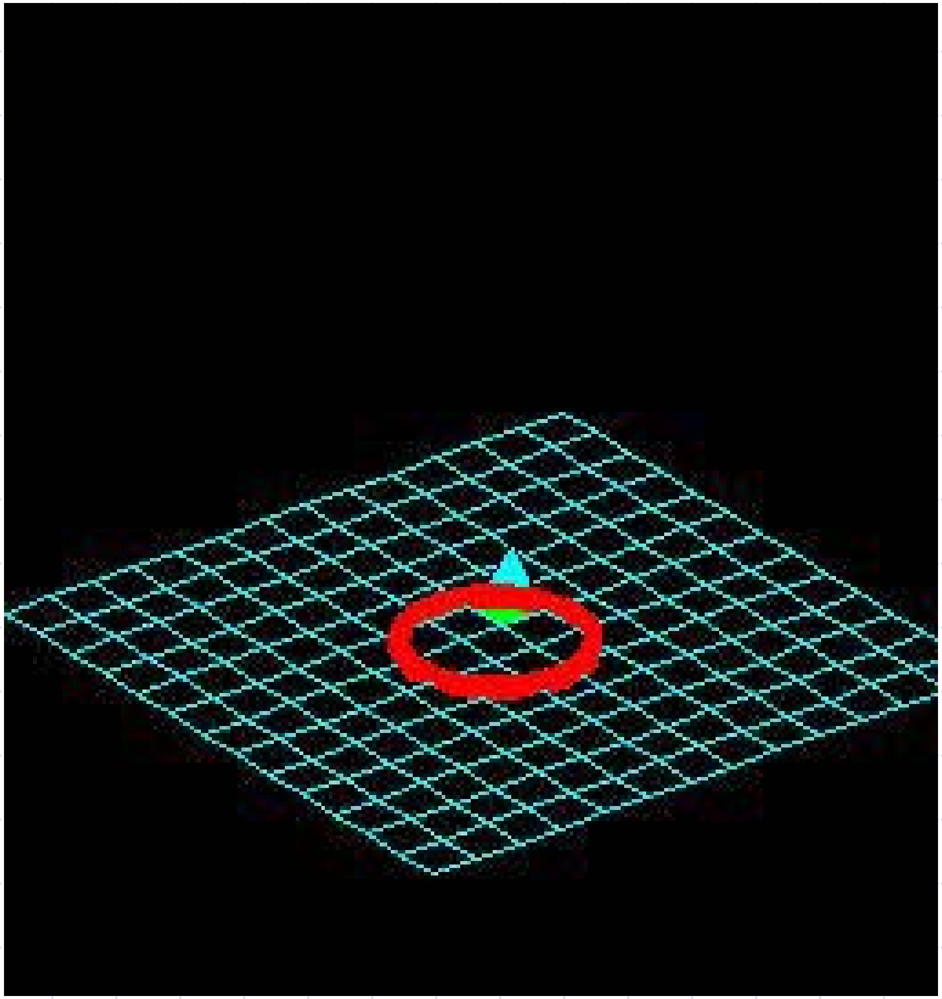




$$dx = F \cdot dT \qquad F\frac{dT}{dx} = 1$$

$$F\,|\nabla T| = 1 \qquad T = 0 \text{ on } \Gamma$$

Construction of stationary level set solution



Summary

Boundary Value Formulation:

$$|\nabla T| F = 1$$

$$Front:$$

$$\Gamma(t) = \{(x,y) : T(x,y) = t\}$$

$$F > 0$$

Initial Value Formulation:

$$\phi_t + F|\nabla \phi| = 0$$

$$Front:$$

$$\Gamma(t) = \{(x,y) : \phi(x,y,t) = 0\}$$

$$F \quad \text{arbitrary}$$

Advantages of these perspectives

- **Unchanged in higher dimensions**
- **Topological changes are handled naturally**
- **Geometric properties are are easily determined**

$$\overline{n} = \frac{\nabla \phi}{\left|\nabla \phi\right|} \quad or \quad \overline{n} = \frac{\nabla T}{\left|\nabla T\right|}$$ normal vector

$$k = \nabla \cdot \frac{\nabla \phi}{\left|\nabla \phi\right|}$$ curvature

- **Both equations can be accurately solved using numerical schemes for hyperbolic conservation laws**

Hamilton-Jacobi equation

Level set equation and stationary equation are particular cases of the more general Hamilton-Jacobi equation:

$$\alpha u_t + H(Du, x) = 0$$

$$H(Du, x) = F\left|\nabla u\right| - (1 - \alpha)$$

Du partial derivatives of u in each variable

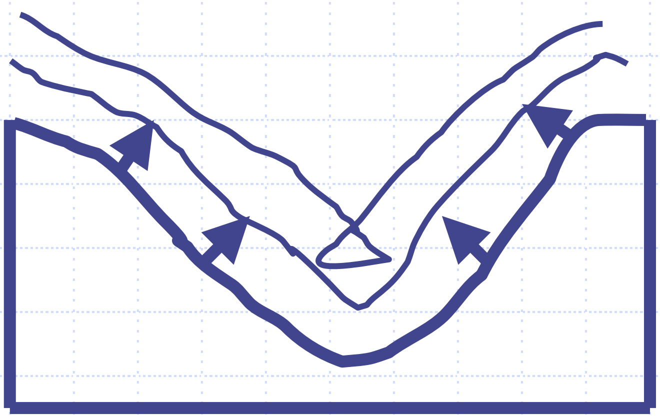
$H(u_x, u_y, u_z, x, y, z)$ Hamiltonian

$\alpha = 1$ level set equation

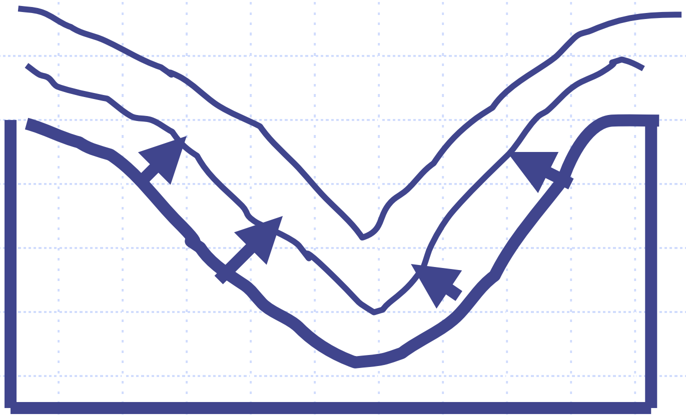
$\alpha = 0$ stationary equation

Example: viscosity solutions

Smooth front, constant speed function F=1



The swallowtail solution



The leading wave solution

Speed function in the form: $F = 1 - \varepsilon \cdot k \quad \varepsilon > 0$

$X^{\varepsilon}_{curvature}(t) \quad , \quad X_{const}(t)$ two solutions, then

$$\lim_{\varepsilon \to 0} X^{\varepsilon}_{curvature}(t) = X_{const}(t)$$

Link between propagating fronts and hyperbolic conservation laws

Hamilton-Jacobi equation with viscosity : $\alpha u_t + H(u_x) = \varepsilon u_{xx}$

Hyperbolic conservation law: $u_t + [G(u)]_x = 0$

Burgers' equation: $u_t + uu_x = 0$

Burgers' equation with viscosity: $u_t + uu_x = \varepsilon u_{xx}$

Conclusion: Level set and Fast marching methods rely on viscosity solutions of the associated partial differential equations in order to guarantee that unique, entropy-satisfying weak solution is obtained. Both equations can be accurately solved using numerical schemes for hyperbolic conservation laws .

Next lectures:

- Efficient numerical algorithms for the Level Set and Fast Marching methods
- Applications of Level Set and Fast Marching methods