

Level Set methods

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Overview

- Introduction

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- Boundary Value Formulation

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- Numerical schemes

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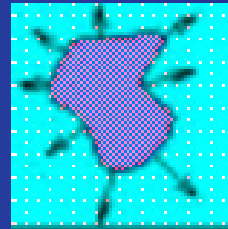
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- Initial Value Formulation
- Numerical schemes
- Fast Marching Methods

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- Narrow Band Level Set

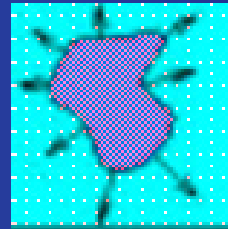
Introduction

moving interface problem



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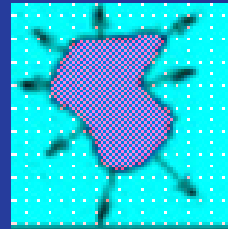
moving interface problem



Speed function $F = F(L, G, I)$

Introduction

moving interface problem

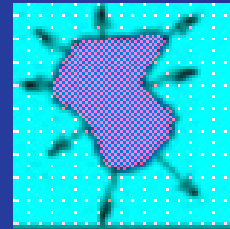


Speed function $F = F(L, G, I)$

Local properties

Introduction

moving interface problem

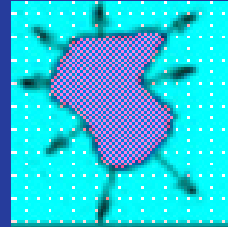


Speed function $F = F(L, G, I)$

Local properties

Global properties

Introduction



moving interface problem

Speed function $F = F(L, G, I)$

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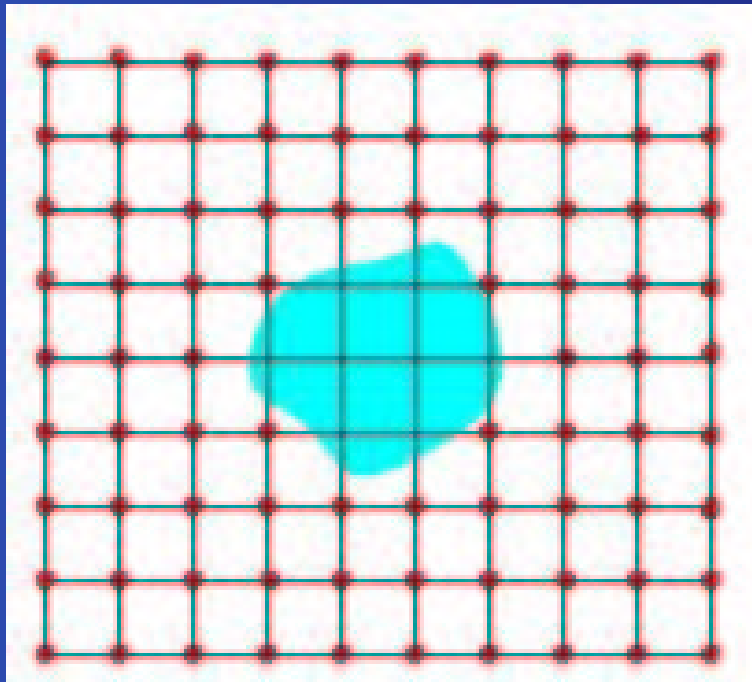
Independent properties

Formulations

- Boundary Value Formulation
- Initial Value Formulation

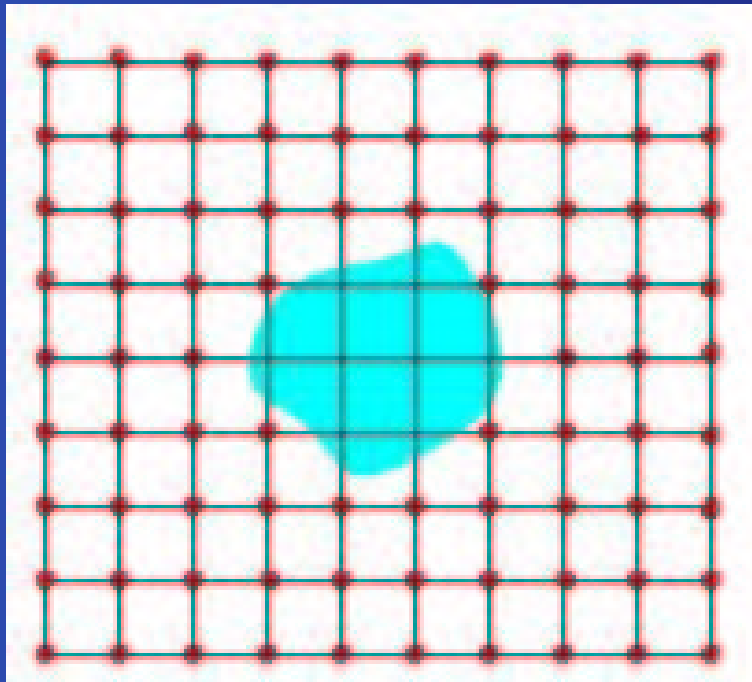
Boundary Value Formulation

Speed function $F > 0$



Boundary Value Formulation

Speed function $F > 0$



Arrival Time T

interface is given by $T(x) = t$

Boundary Value Formulation

distance = rate · time

$$1 = F \frac{dT}{dx}$$

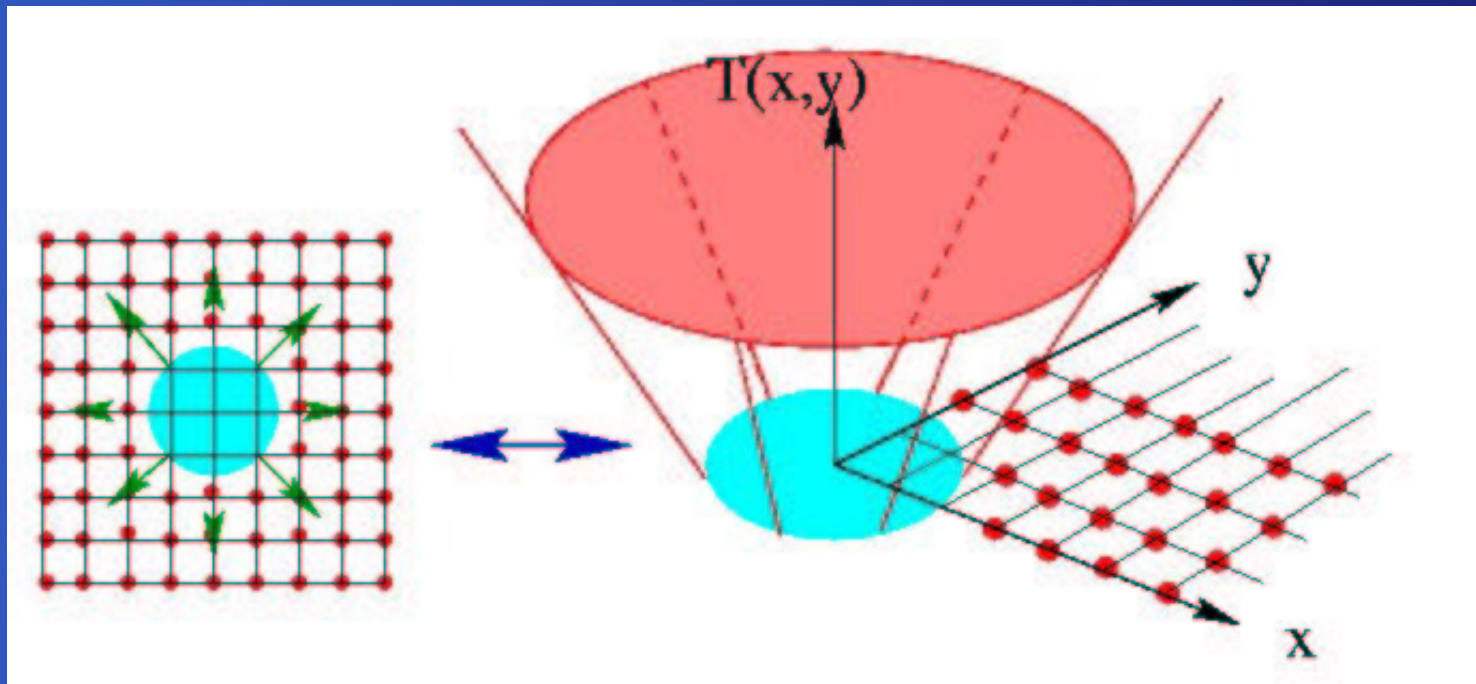
More dimensions:

$$|\nabla T| F = 1$$

with $T = 0$ on Γ

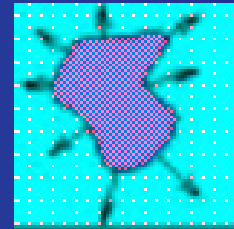
Example Boundary Value Formulation

Γ is circular, $F = 1$:



Initial Value Formulation

Level Set Function $\phi(x(t), t)$



Interface is given by $\phi(x(t), t) = 0$.

Initial Value Formulation

level set function $\phi(x(t), t) = 0$

By the chain rule:

$$\phi_t + \nabla \phi(x(t), t) \cdot x'(t) = 0$$

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F is speed in outward normal direction, then
 $x'(t) \cdot n = F, n = \frac{\nabla\phi}{|\nabla\phi|}$

Initial Value Formulation

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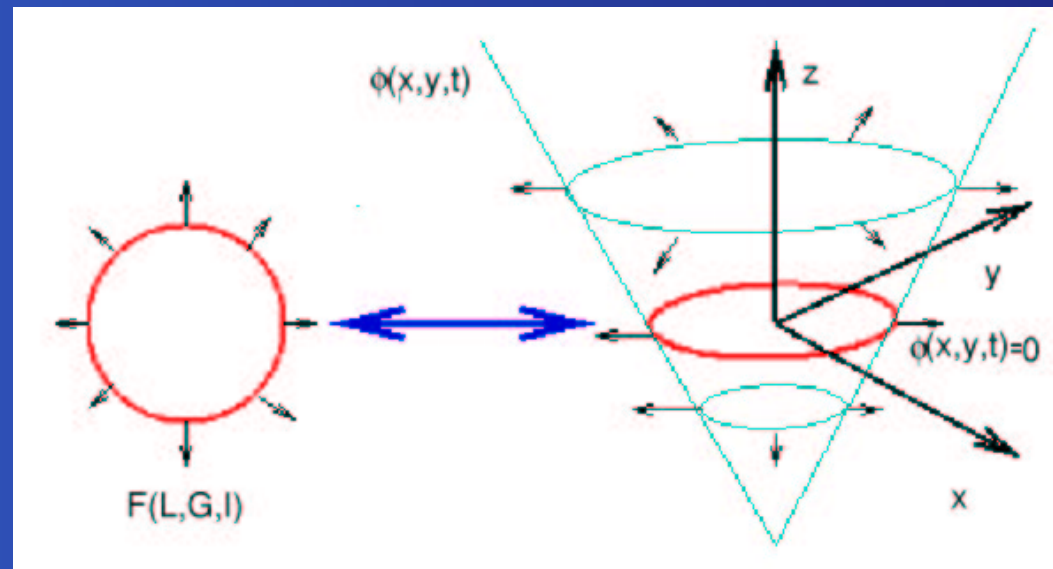
F is speed in outward normal direction, then

$$x'(t) \cdot n = F, \quad n = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\phi_t + F|\nabla\phi| = 0$$

$$\text{given } \phi(x, t = 0)$$

Example Initial Value Formulation



Summary

Boundary Value Formulation ($F > 0$)

$$|\nabla T|F = 1$$

$$\text{Front} = \Gamma(t) = \{(x, y) | T(x, y) = t\}$$

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Initial Value Formulation

$$\phi_t + F|\nabla\phi| = 0$$

$$\text{Front} = \Gamma(t) = \{(x, y) | \phi(x, y, t) = 0\}$$

Numerical Scheme for BVP

$$|\nabla T|F(x) = 1$$

$$\text{front: } \Gamma = \{x | T(x) = 0\}$$

Numerical Scheme for BVP

$$|\nabla T| F(x) = 1$$

$$\text{front: } \Gamma = \{x | T(x) = 0\}$$

$$\left[\max(D_i^{-x}T, 0)^2 + \min(D_i^{+x}T, 0)^2 \right]^{\frac{1}{2}} = \frac{1}{F_i}$$

where $D_i^{-x}T = \frac{T_i - T_{i-1}}{\Delta x}$ Backward Scheme
and $D_i^{+x}T = \frac{T_{i+1} - T_i}{\Delta x}$ Forward Scheme.

Numerical Scheme for IVP

$$\phi_t + F|\nabla\phi| = 0$$

$$\text{front: } \Gamma = \{x | \phi(x, t) = 0\}$$

Numerical Scheme for IVP

$$\phi_t + F|\nabla\phi| = 0$$

$$\text{front: } \Gamma = \{x | \phi(x, t) = 0\}$$

$$\phi_i^{n+1} = \phi_i^n - \Delta t [\max(F_i, 0)\nabla^+ + \min(F_i, 0)\nabla^-]$$

where

$$\nabla^+ = [\max(D_i^{-x}, 0)^2 + \min(D_i^{+x}, 0)^2]^{\frac{1}{2}}$$

$$\nabla^- = [\max(D_i^{+x}, 0)^2 + \min(D_i^{-x}, 0)^2]^{\frac{1}{2}}$$

CFL condition

Time Step restriction

$$\max F \Delta t \leq \Delta x$$

Notes

- Extensions to higher order schemes

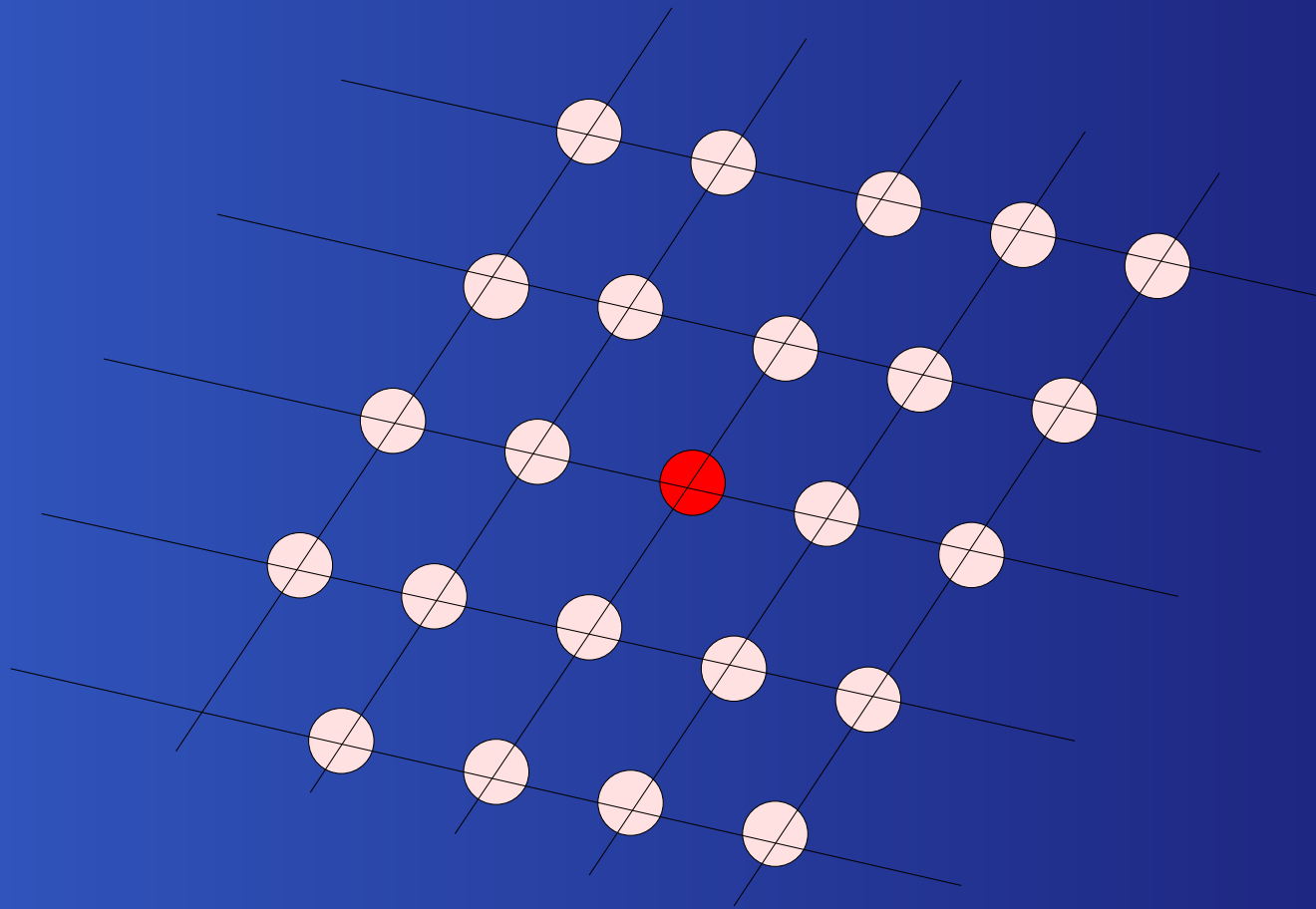
Notes

- Extensions to higher order schemes
- Schemes for non-convex speed function

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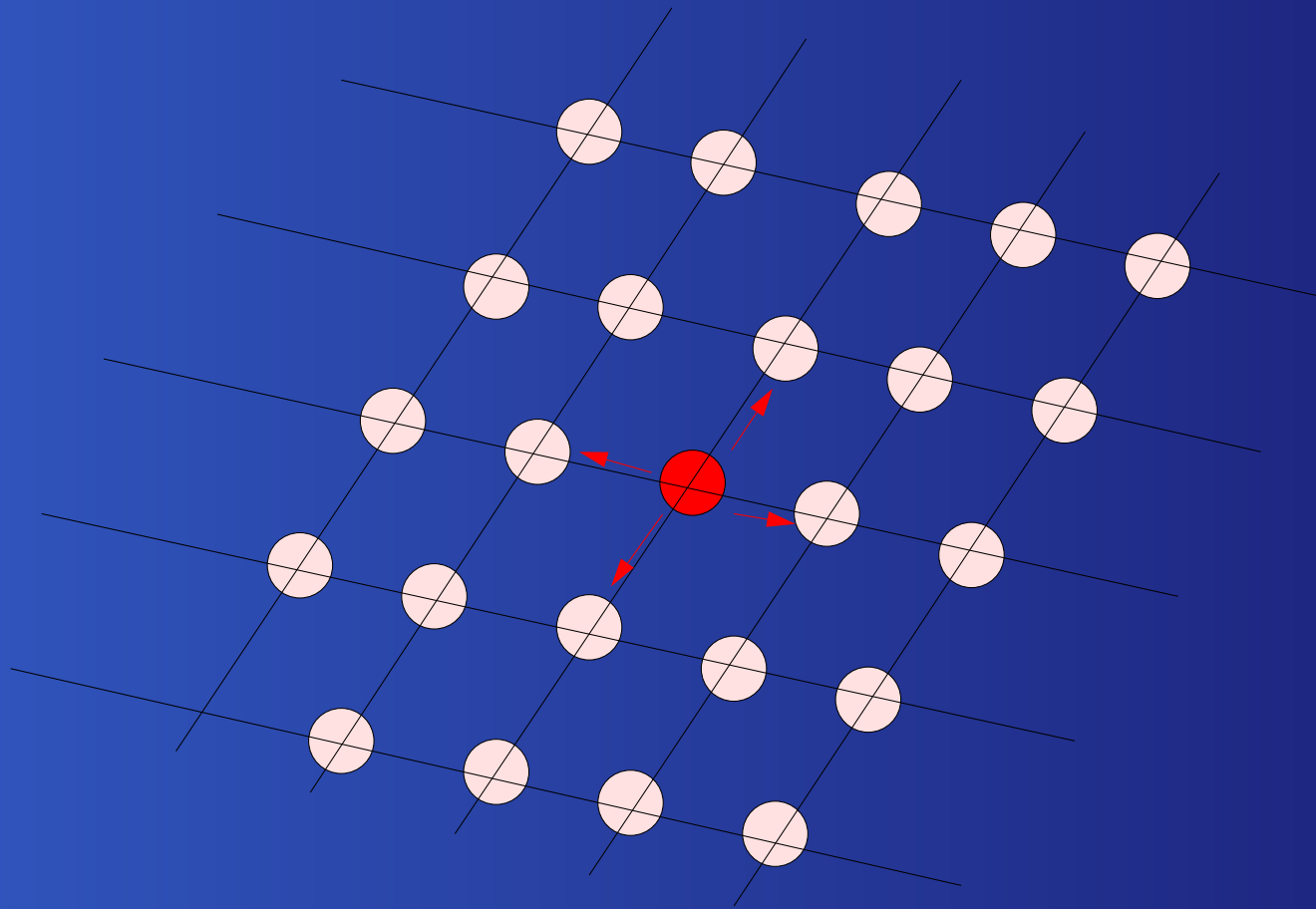
- Extensions to higher order schemes
- Schemes for non-convex speed function
- These Numerical Schemes costs a lot of computations

Fast Marching Method



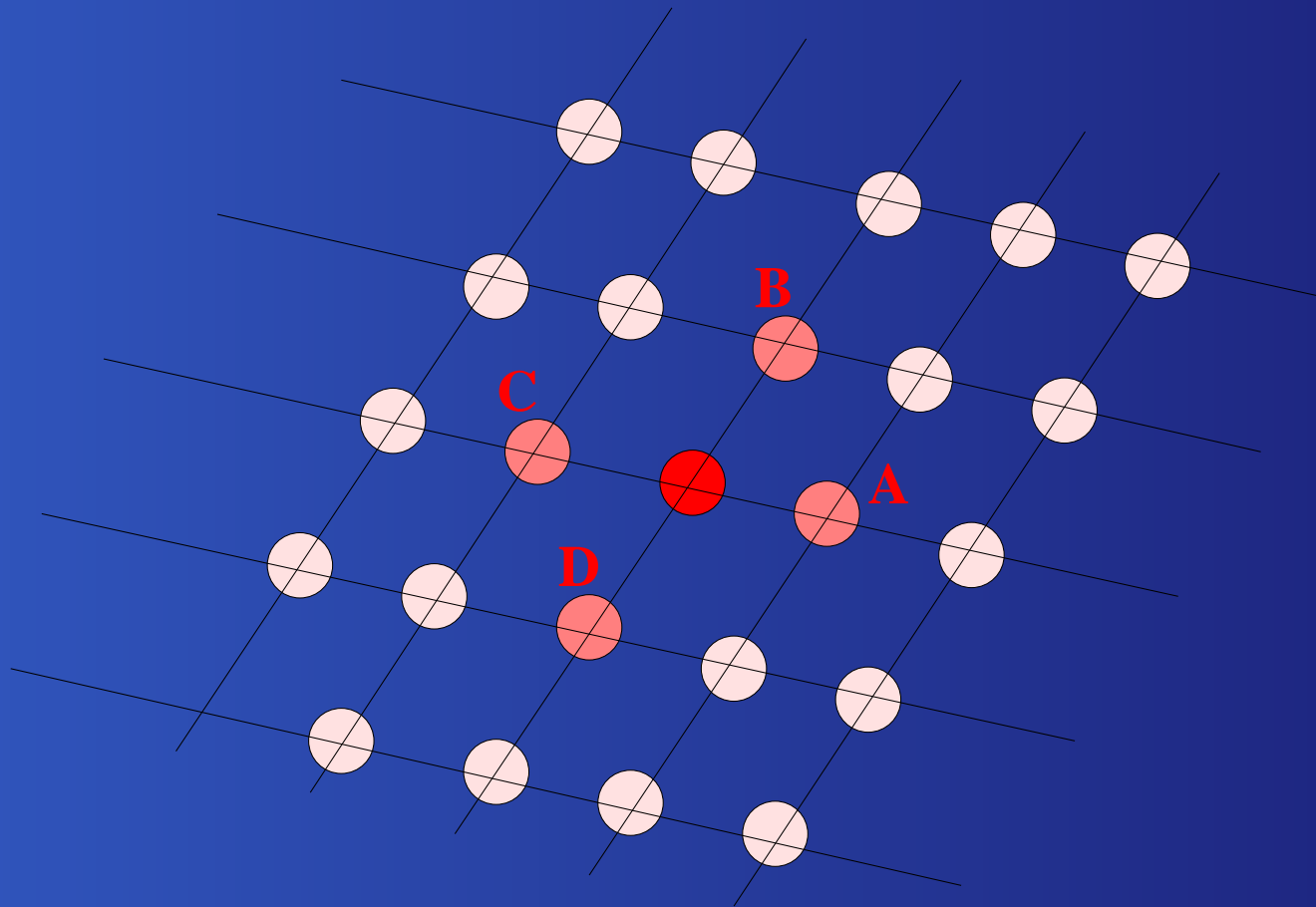
Beginning of Fast Marching Method

Fast Marching Method



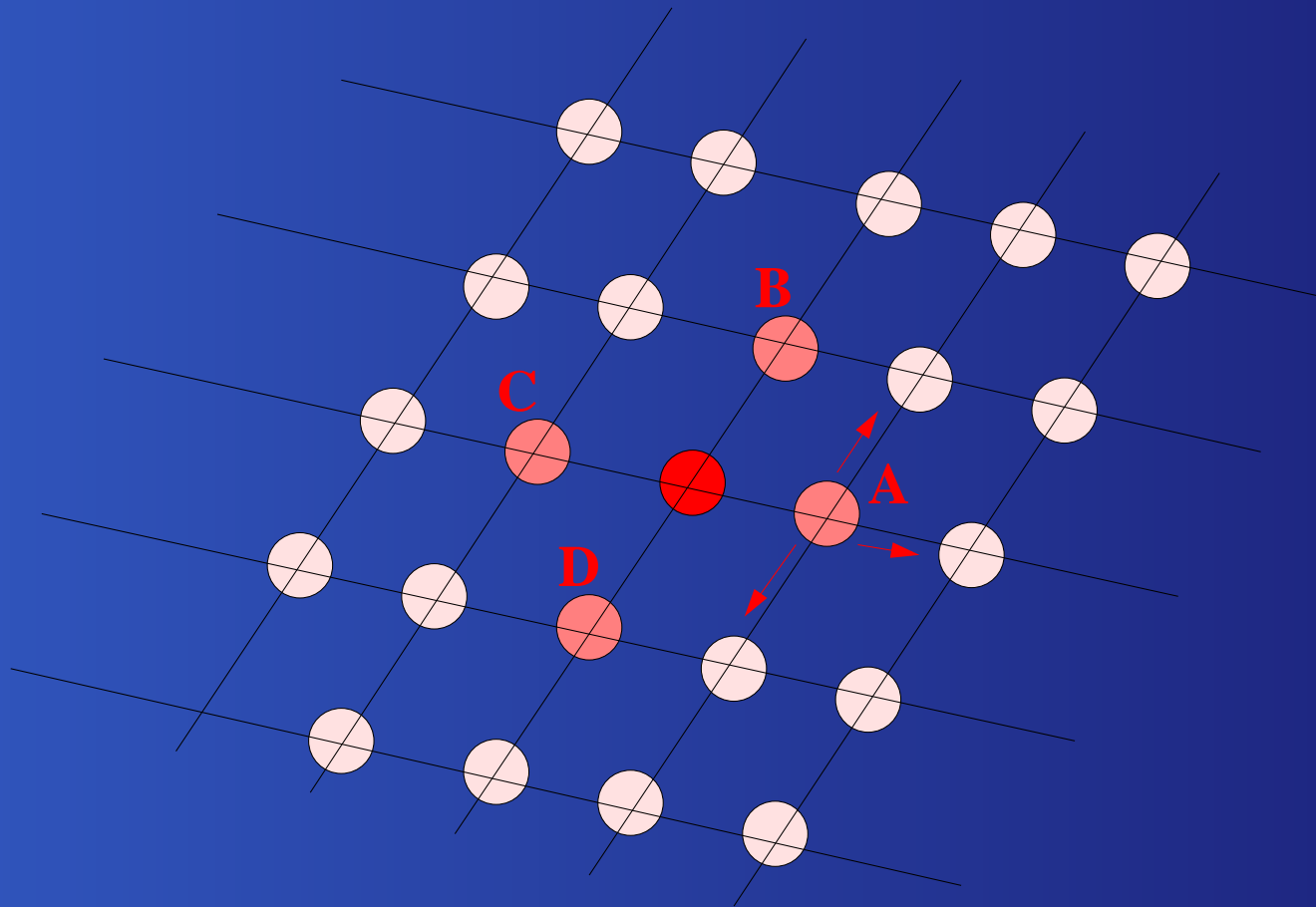
Update “downwind”

Fast Marching Method



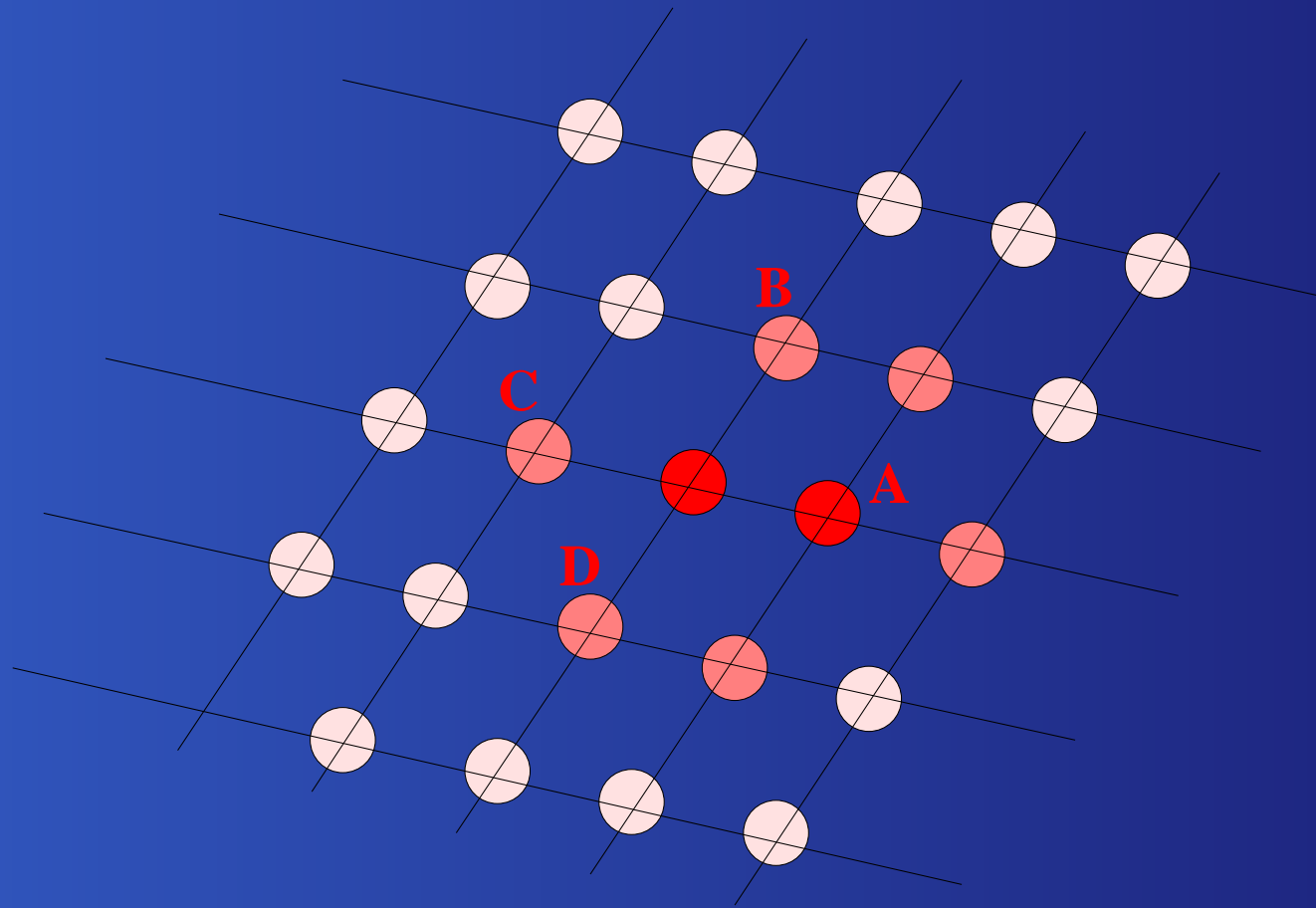
Compute new possible values

Fast Marching Method



Choose smallest dark pink sphere (A)

Fast Marching Method



Freeze value at A, update neighboring downwind points

Fast Marching Methods

Initialisation: Tag points as **Known**, **Trial** and **Far** .

Fast Marching Methods

- Begin Loop: Let A be the **Trial** point with the smallest T value

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- Tag as **Trial** all neighbors of A that are not **Known** . If neighbor is in **Far** , remove and add to the set **Trial** .

Fast Marching Methods

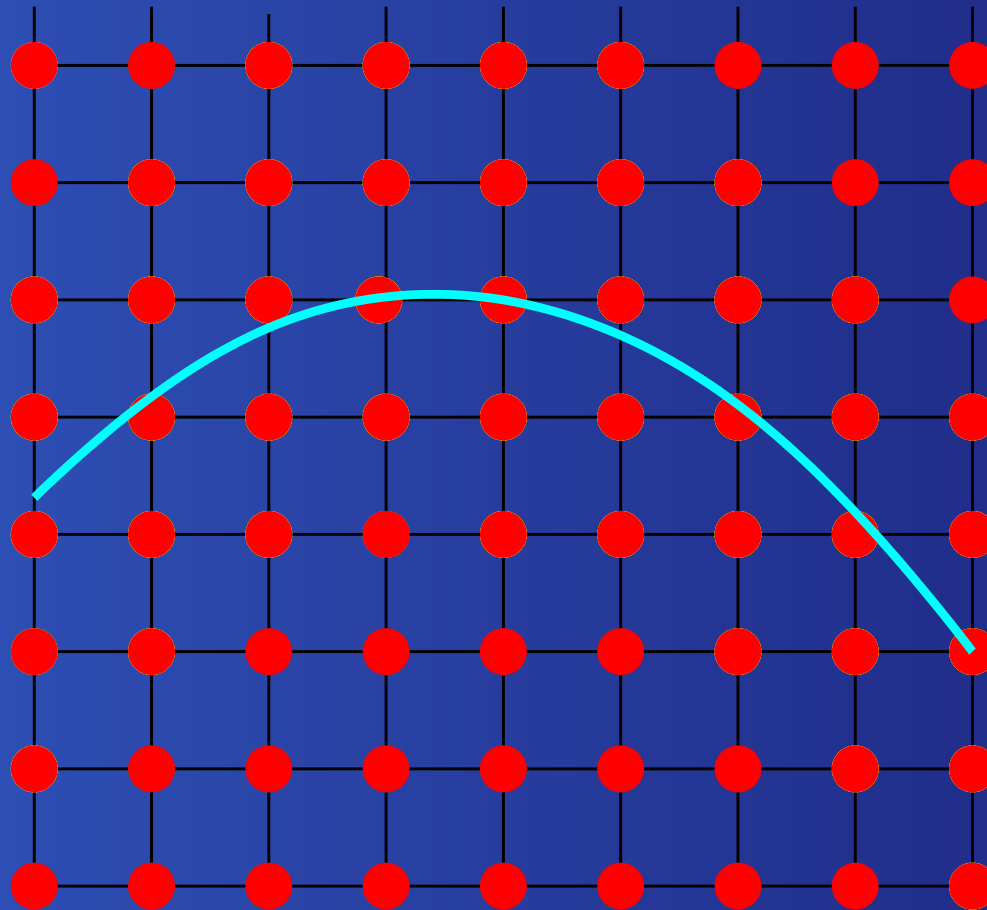
- Recompute the values of T at all **Trial** neighbors of A according to

$$\left[\max \left(D_{ijk}^{-x} T, -D_{ijk}^{+x} T, 0 \right)^2 + \max \left(D_{ijk}^{-y} T, -D_{ijk}^{+y} T, 0 \right)^2 + \max \left(D_{ijk}^{-z} T, -D_{ijk}^{+z} T, 0 \right)^2 \right]^{\frac{1}{2}} = \frac{1}{F_{ijk}}$$

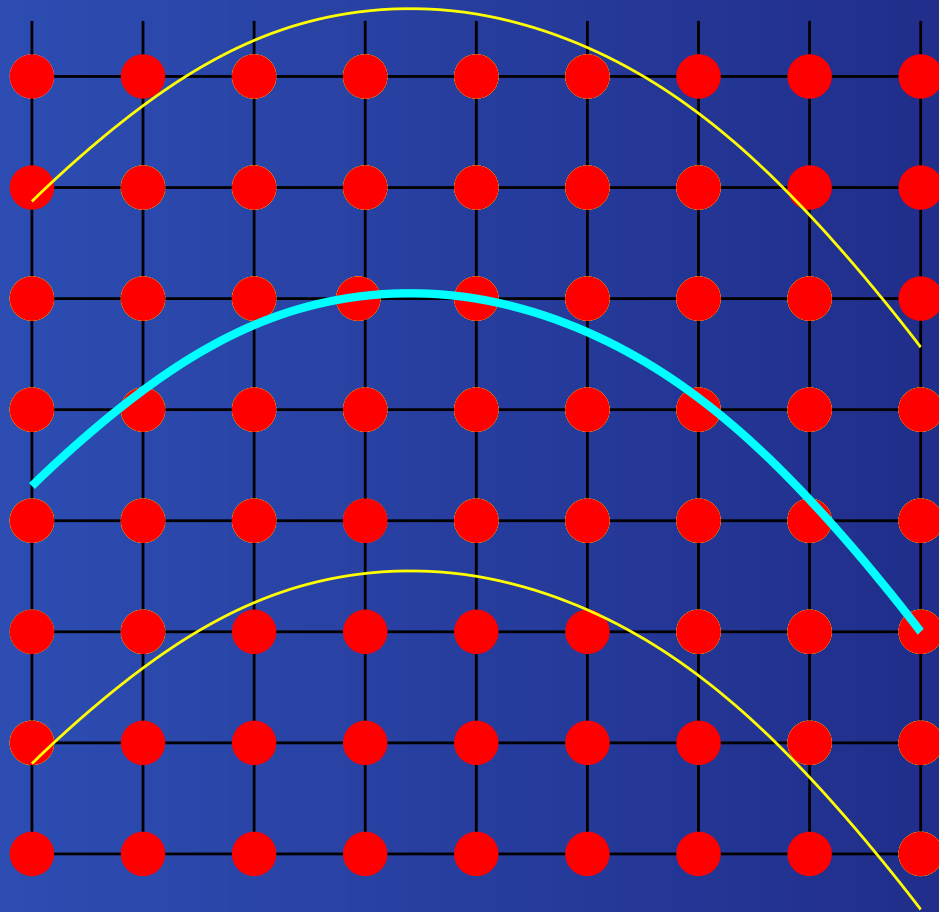
Fast Marching Methods

- Begin Loop: Let A be the **Trialpoint** with the smallest T value
- Add the point A to **Known** ; remove it from **Trial**
- Tag as **Trial** all neighbors of A that are not **Known** . If neighbor is in **Far** , remove and add to the set **Trial** .
- Recompute the values of T at all **Trial** neighbors of A according to ...
- Return to top of loop.

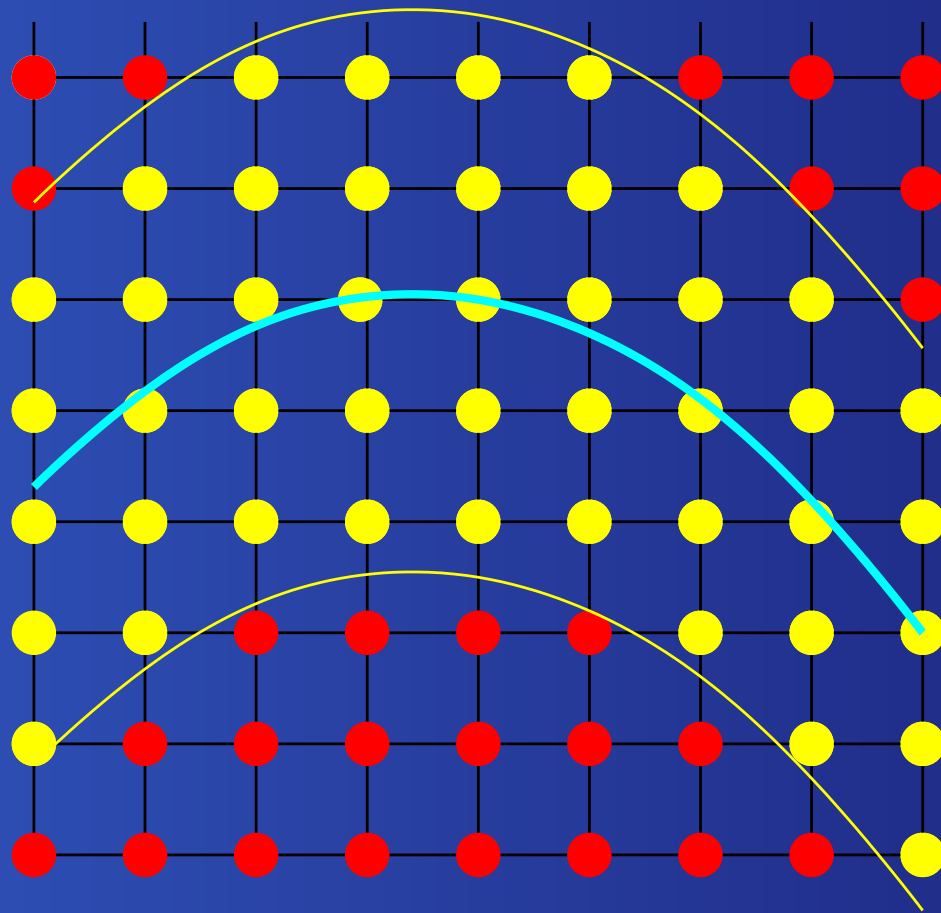
Narrow Band Level Set



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Narrow Band Level Set

- Tag **Alive** points in narrow band

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Narrow Band Level Set

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- Initialize **Far Away** points outside the narrow band with large positive (negative) values if values are outside (inside) the front itself

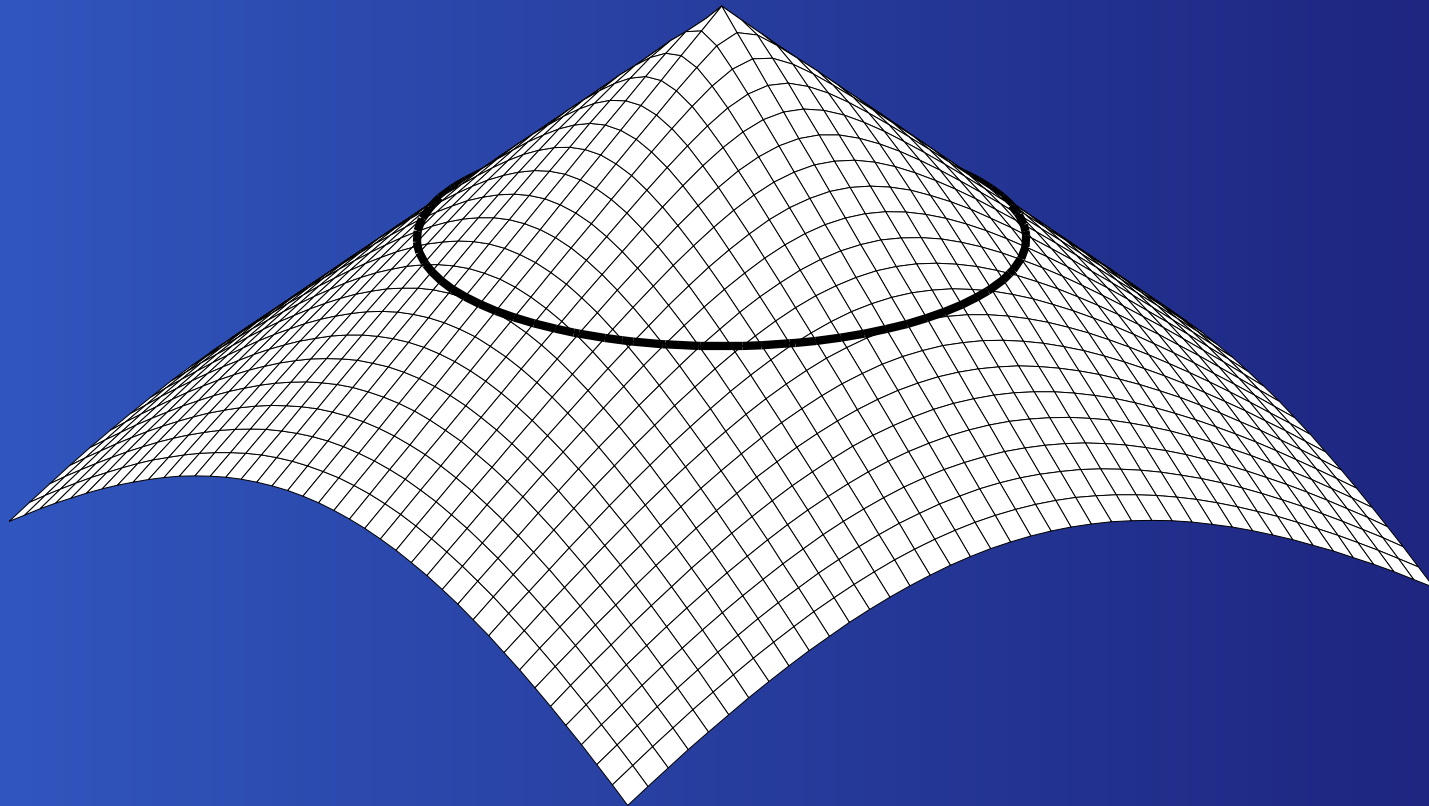
Narrow Band Level Set

- Tag **Alive** points in narrow band
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- Solve level set equation until **Land Mine** hit

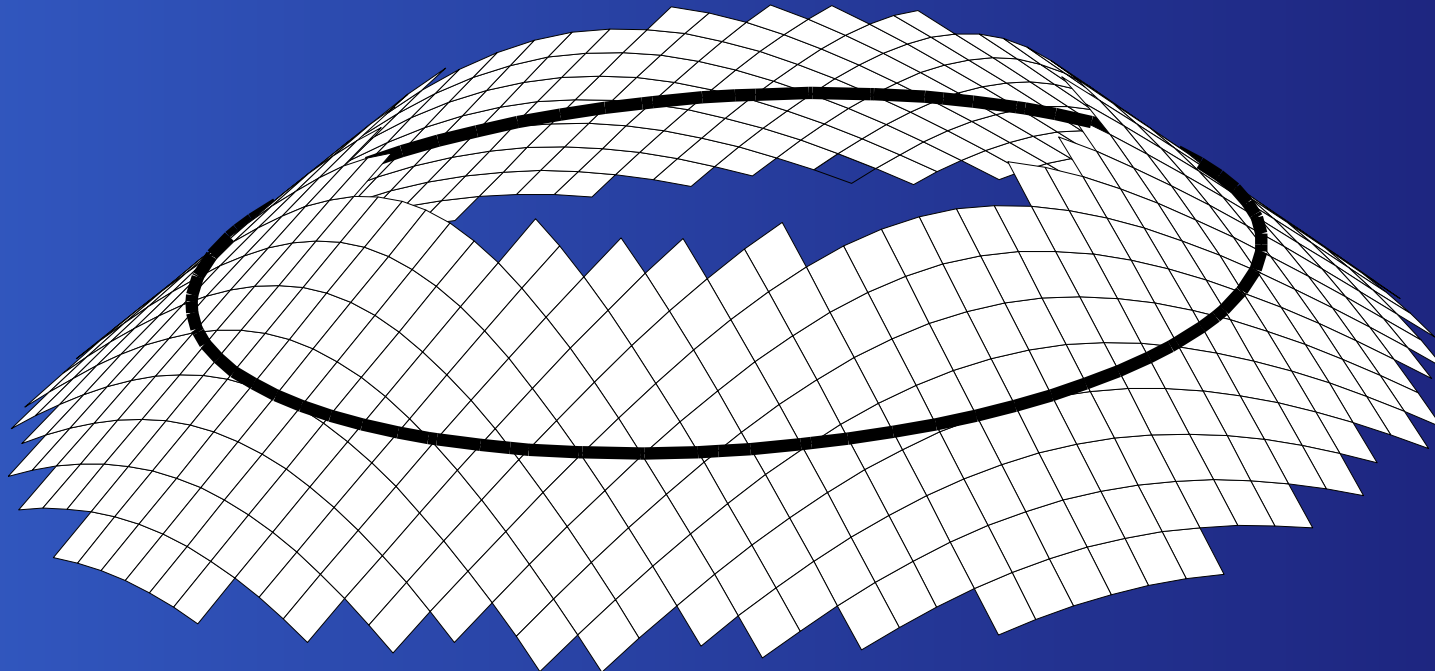
Narrow Band Level Set

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- Solve level set equation until **Land Mine** hit
- Rebuild and loop

Narrow Band Level Set



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Advantages of Narrow Band Level Set

- Speed

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Advantages of Narrow Band Level Set

- Speed
 - $O(kN^2)$ instead of $O(N^3)$ in 3D
- Timestep
 - CFL condition on narrow band instead of entire domain

$$\max F \Delta T \leq \Delta x$$

Next lecture

Applications

End