technische universiteit eindhoven

Seminar

B-splines in Geometric Modeling

June 9th, 2004

W. Dijkstra

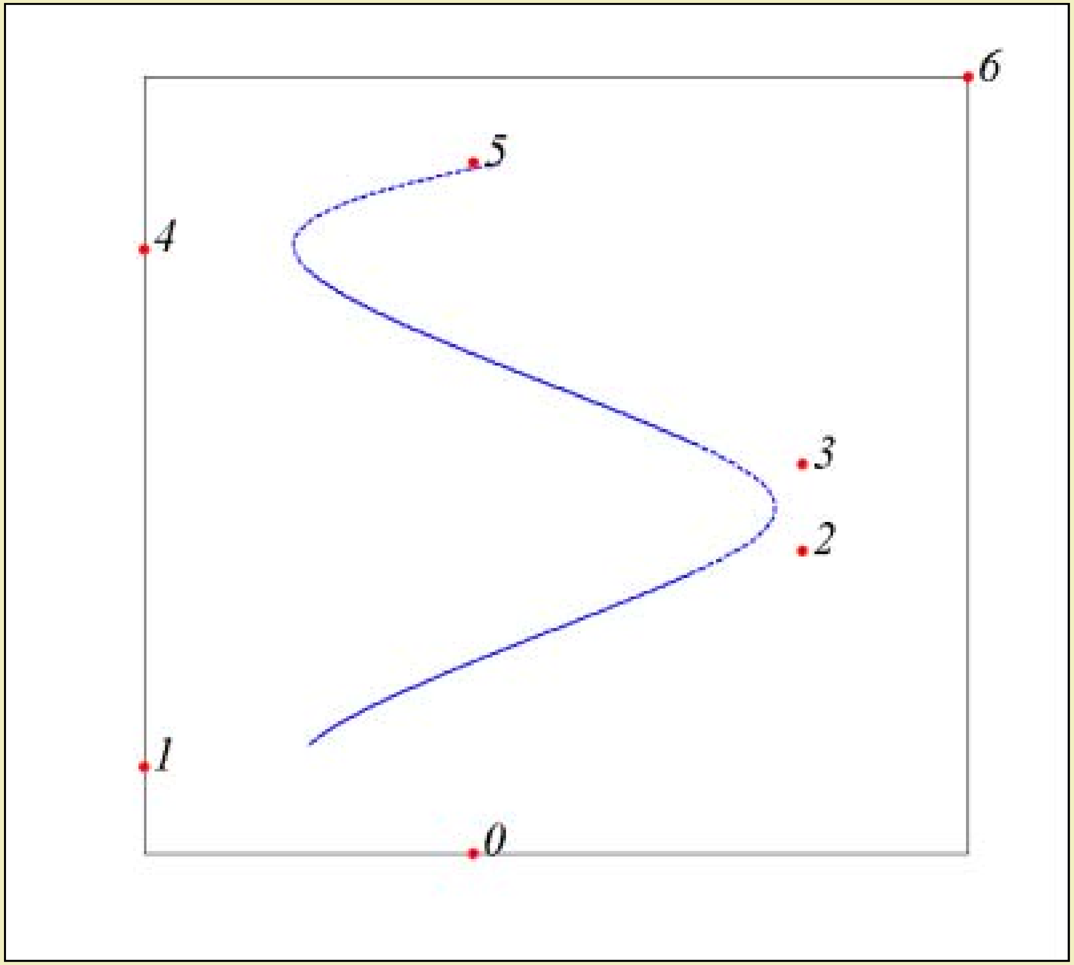
Scientific Computing Group

Overview

- 09-06: B-spline curves (W. Dijkstra)
- 16-06: NURBS (E. Shcherbakov)
- 30-06: B-spline surfaces (M. Patricio)

Introduction

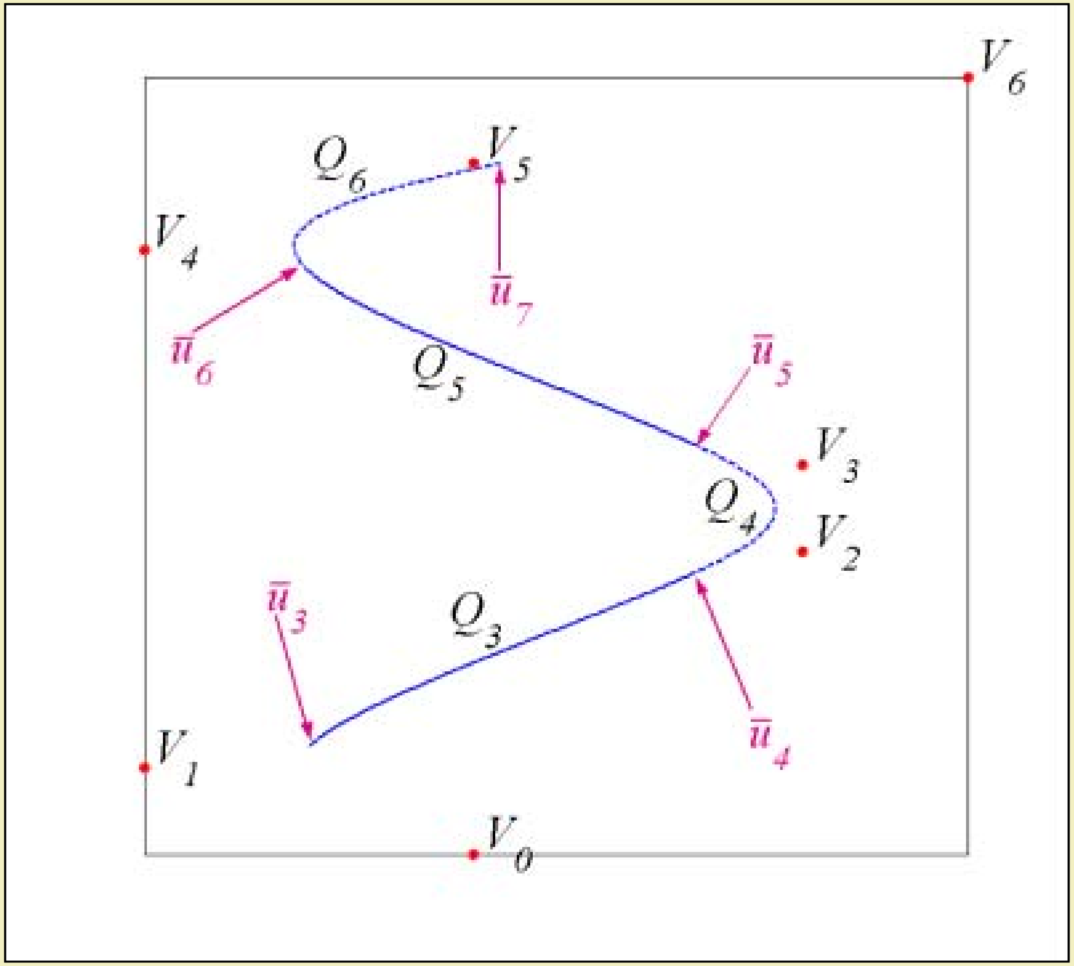
- Set of points
- Curve passing near points



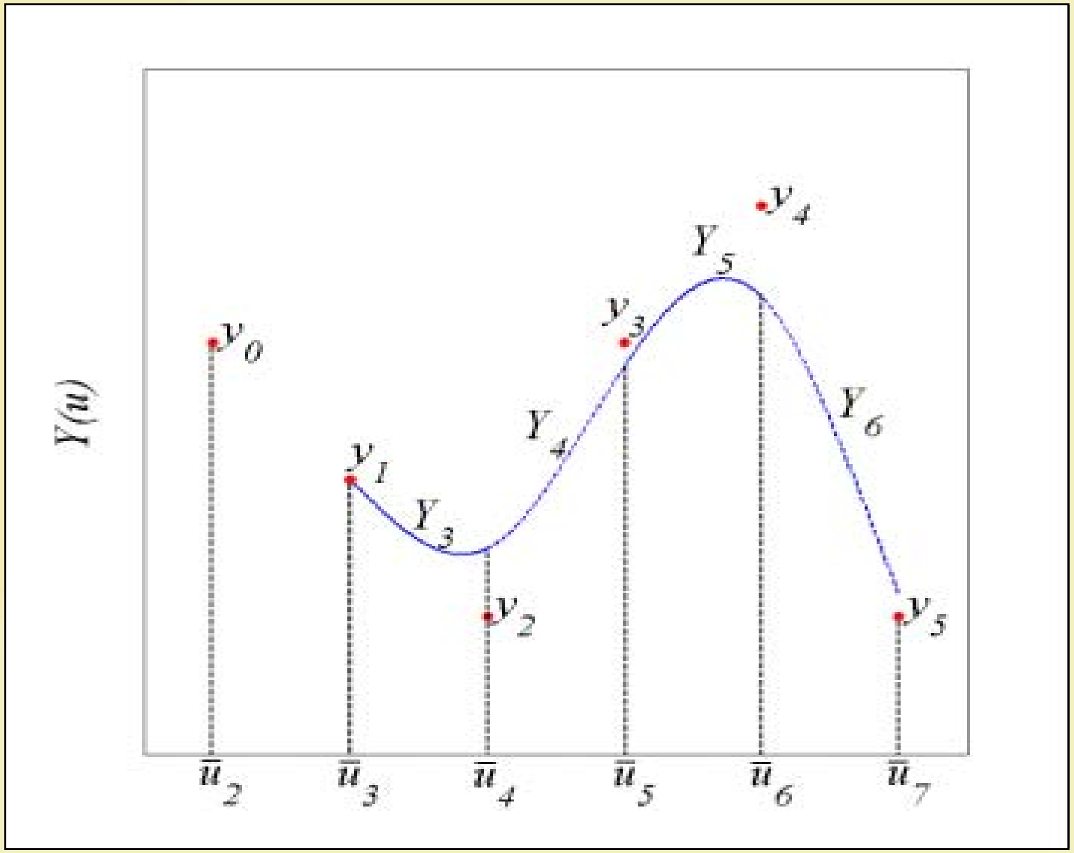
Definitions

- Vertices $V_1,\ldots,V_m$
- Curve segments

  $Q_i = (X_i, Y_i)$
- Knots $\overline{u}_i$
- Global parameter: $\overline{u}$

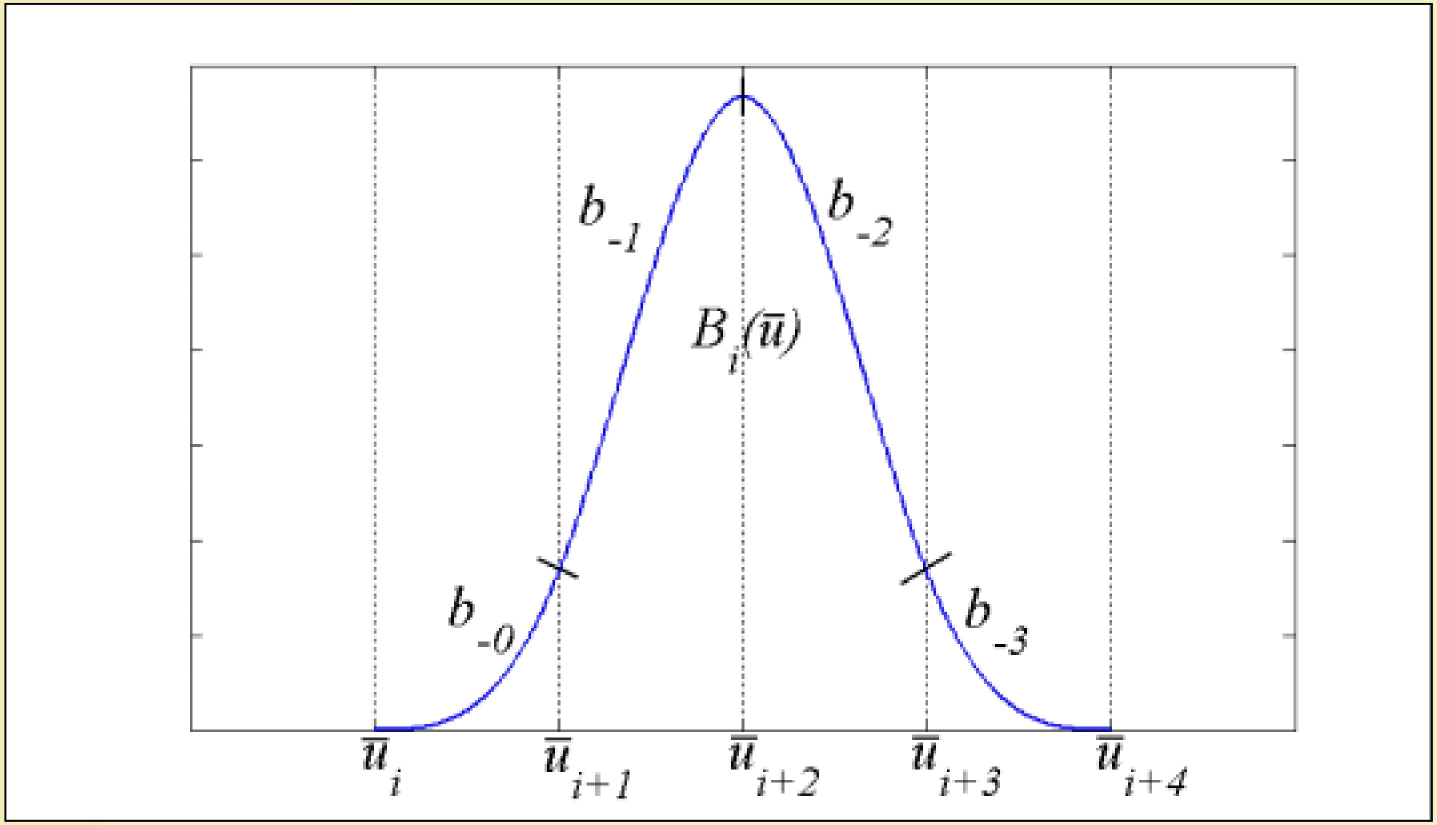


Y-Coordinate



$$Y_i(\bar{u}) = y_{i-3}B_{i-3}(\bar{u}) + y_{i-2}B_{i-2}(\bar{u}) + y_{i-1}B_{i-1}(\bar{u}) + y_{i-0}B_{i-0}(\bar{u})$$

Basis function B_i



$$b_j(u) = a_j + b_j u + c_j u^2 + d_j u^3, \quad 0 \le u \le 1$$

Conditions

- Continuous in position, and 1st and 2nd derivative in knots

$$
\begin{aligned}
&0^{\text{th}}: \quad b_{-0}(1) = b_{-1}(0), \quad b_{-1}(1) = b_{-2}(0), \quad b_{-2}(1) = b_{-3}(0) \\
&1^{\text{st}}: \quad b_{-0}{}^{(1)}(1) = b_{-1}{}^{(1)}(0), \quad b_{-1}{}^{(1)}(1) = b_{-2}{}^{(1)}(0), \quad b_{-2}{}^{(1)}(1) = b_{-3}{}^{(1)}(0) \qquad (9) \\
&2^{\text{nd}}: \quad b_{-0}{}^{(2)}(1) = b_{-1}{}^{(2)}(0), \quad b_{-1}{}^{(2)}(1) = b_{-2}{}^{(2)}(0), \quad b_{-2}{}^{(2)}(1) = b_{-3}{}^{(2)}(0)
\end{aligned}
$$

- Zero position, 1st and 2nd derivative in start and end points

$$
\begin{aligned}
&0^{\text{th}}: \quad b_{-0}(0) = 0, \quad b_{-3}(1) = 0 \\
&1^{\text{st}}: \quad b_{-0}{}^{(1)}(0) = 0, \quad b_{-3}{}^{(1)}(1) = 0 \qquad (6) \\
&2^{\text{nd}}: \quad b_{-0}{}^{(2)}(0) = 0, \quad b_{-3}{}^{(2)}(1) = 0
\end{aligned}
$$

Conditions

- Sum equal to 1

$$b_{-0}(0)+b_{-1}(0)+b_{-2}(0)+b_{-3}(0)=1$$

we already have: $b_{-3}(0)=0$

$$b_{-0}(0)+b_{-1}(0)+b_{-2}(0)=1$$

(1)

+ ———

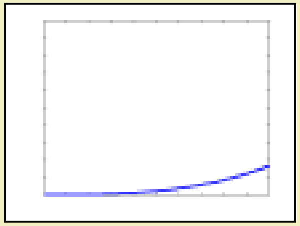
(16)

16 conditions

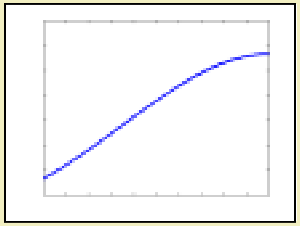
16 coefficients a_j, b_j, c_j, d_j

Cubical polynomials

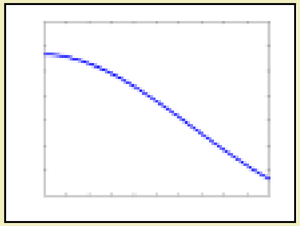
$$b_{-0}(u) = \frac{1}{6}u^3$$



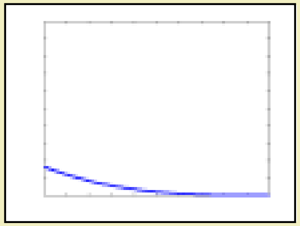
$$b_{-1}(u) = \frac{1}{6}\left(1 + 3u + 3u^2 - 3u^3\right)$$



$$b_{-2}(u) = \frac{1}{6}\left(4 - 6u^2 + 3u^3\right)$$



$$b_{-3}(u) = \frac{1}{6}\left(1 - 3u + 3u^2 - u^3\right)$$



It can be shown that:

$$b_{-0}(u) + b_{-1}(u) + b_{-2}(u) + b_{-3}(u) = 1$$

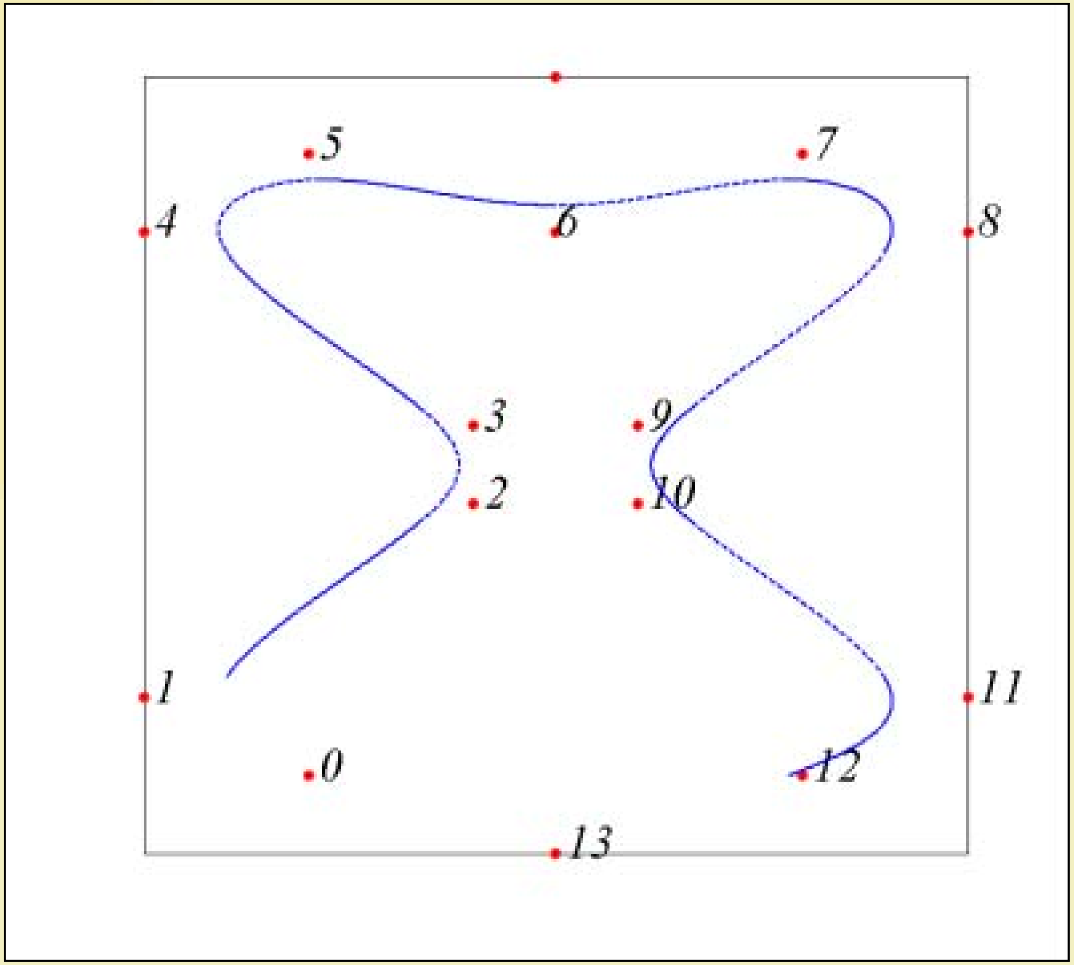
B-splines

Basic function: $$B_i(\bar{u}) = \begin{cases} b_{-0}(u), & \bar{u}_i \le \bar{u} \le \bar{u}_{i+1} \\ b_{-1}(u), & \bar{u}_{i+1} \le \bar{u} \le \bar{u}_{i+2} \\ b_{-2}(u), & \bar{u}_{i+2} \le \bar{u} \le \bar{u}_{i+3} \\ b_{-3}(u), & \bar{u}_{i+3} \le \bar{u} \le \bar{u}_{i+4} \end{cases}$$

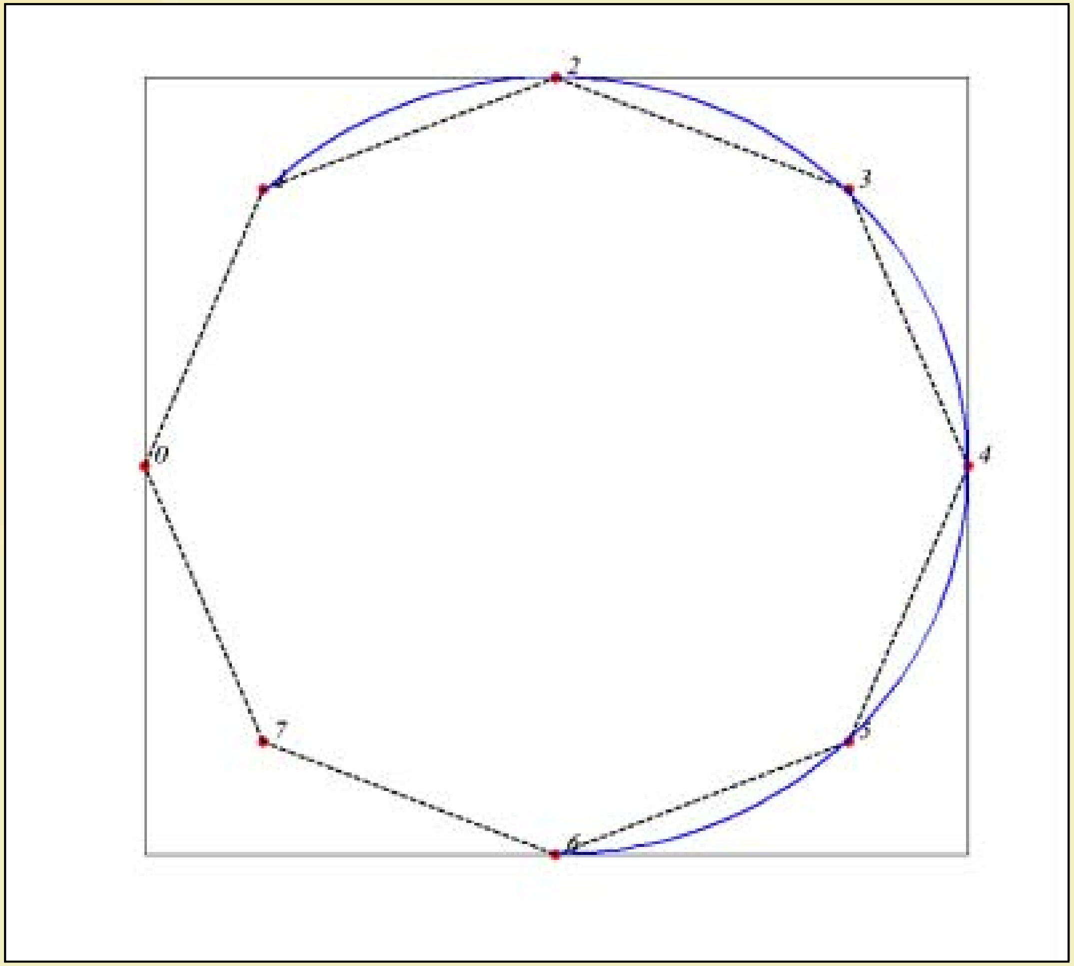
B-spline in segment *i*:

$$Y_i(\bar{u}) = y_{i-3}B_{i-3}(\bar{u}) + y_{i-2}B_{i-2}(\bar{u}) + y_{i-1}B_{i-1}(\bar{u}) + y_{i-0}B_{i-0}(\bar{u})$$

Example



Convex hull



Properties

Invariant under:

- Translation
- Rotation
- Scaling

Double/triple end vertices

- Double:

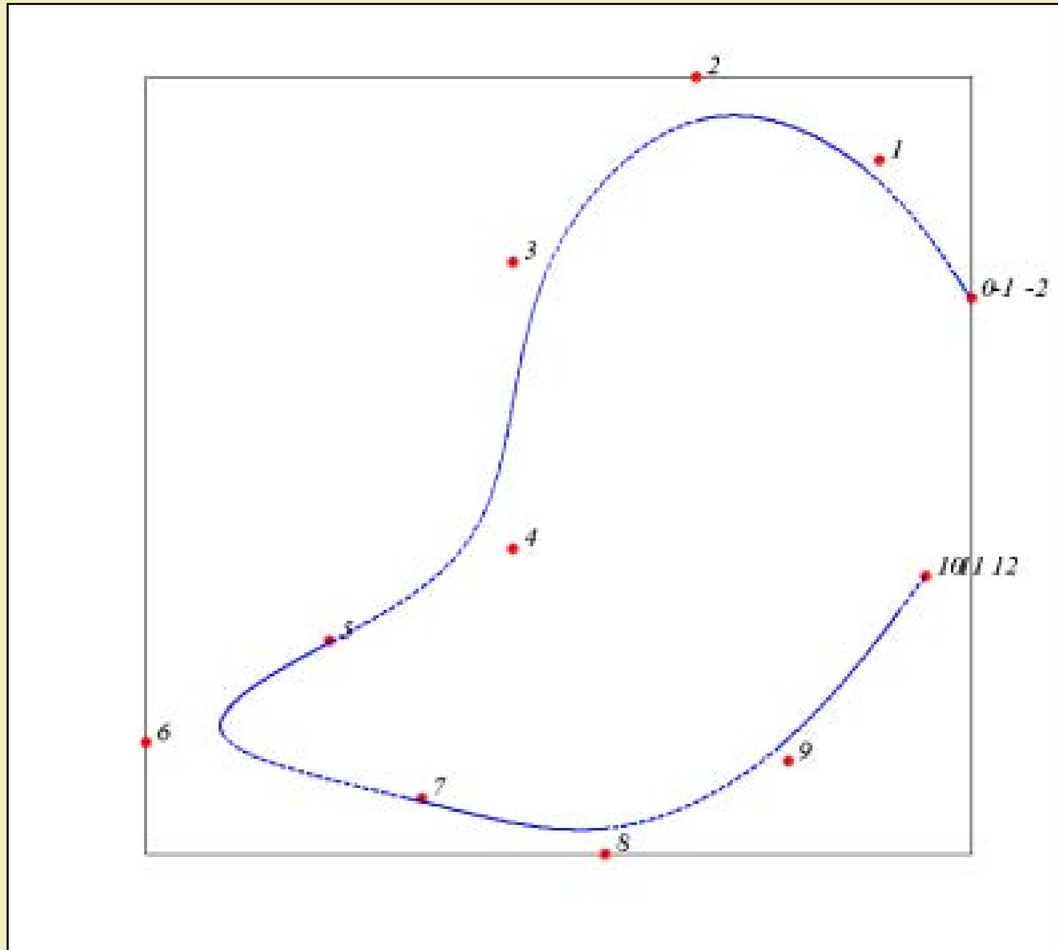
$$\mathbf{V}_{-1} = \mathbf{V}_0$$

$$\mathbf{V}_{m+1} = \mathbf{V}_m$$

- Triple:

$$\mathbf{V}_{-2} = \mathbf{V}_{-1} = \mathbf{V}_0$$

$$\mathbf{V}_{m+2} = \mathbf{V}_{m+1} = \mathbf{V}_m$$

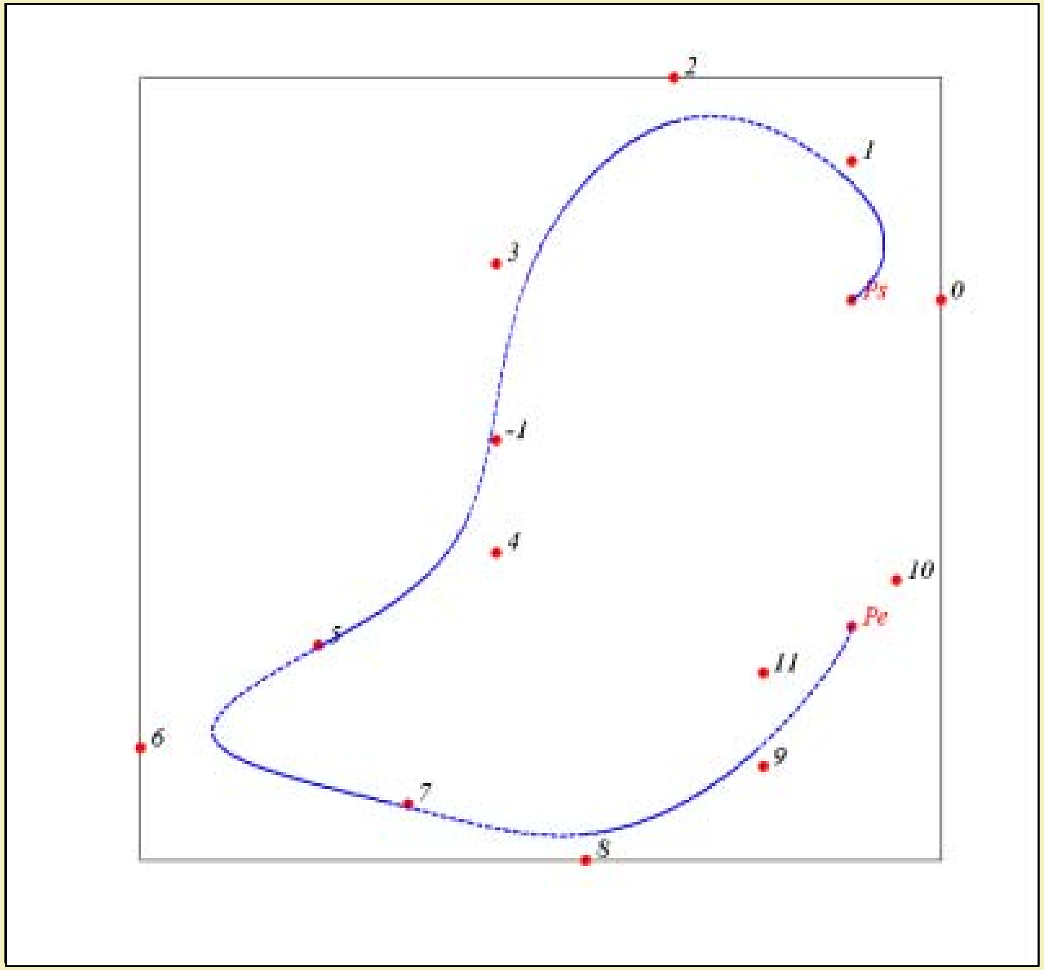


End vertex specification

- Choose P_s and P_e
- Add vertices:

$$\mathbf{V}_{-1} = 6\mathbf{P}_s - 4\mathbf{V}_0 - \mathbf{V}_1$$

$$\mathbf{V}_{m+1} = 6\mathbf{P}_e - 4\mathbf{V}_m - \mathbf{V}_{m-1}$$

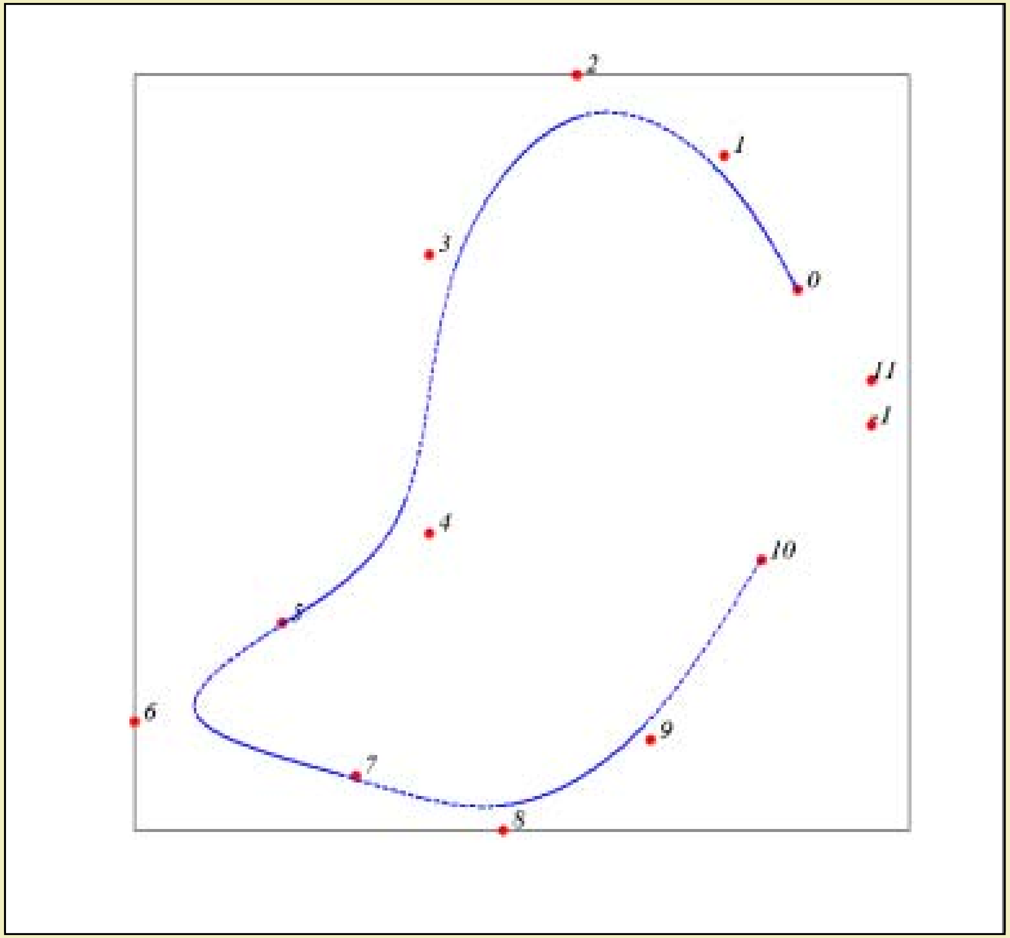


End vertex interpolation

- Curve must go through V_0 and V_m
- Add vertices:

$$\mathbf{V}_{-1} = 2\mathbf{V}_0 - \mathbf{V}_1$$

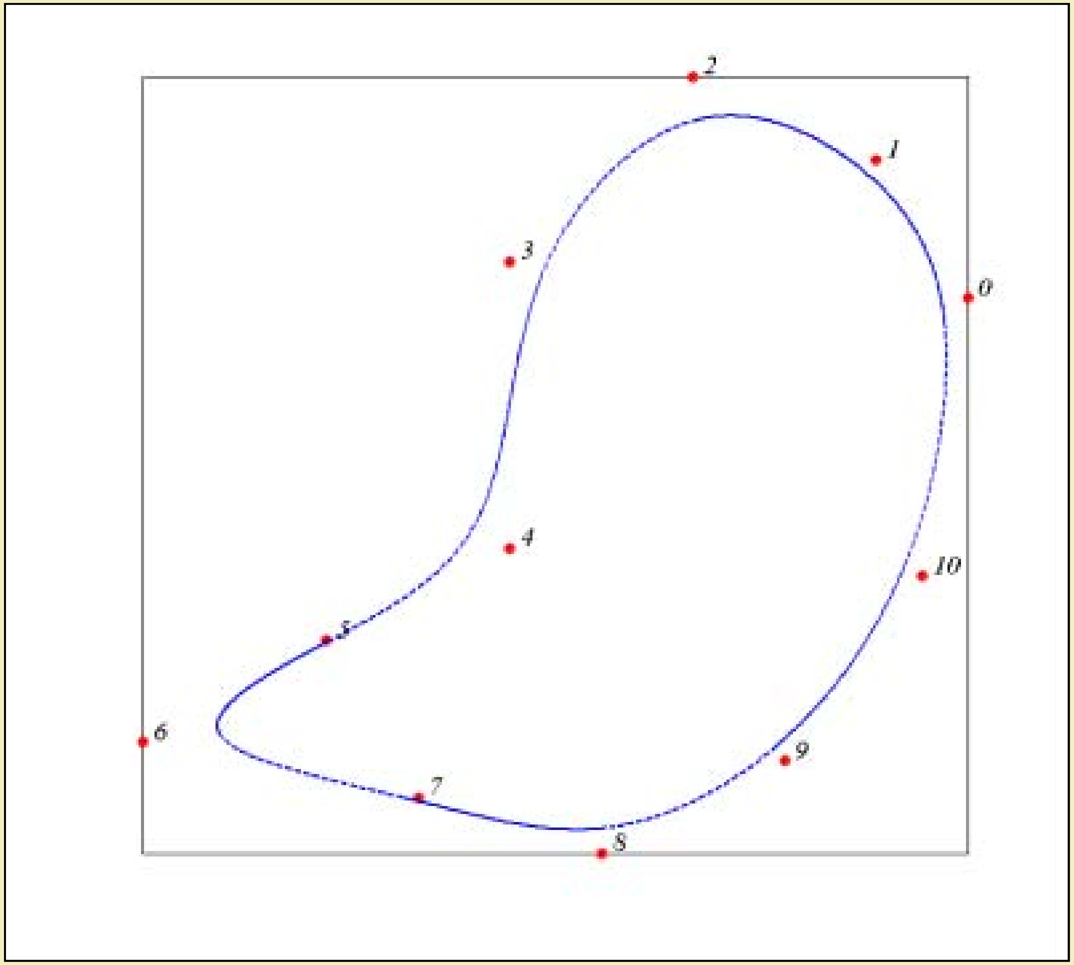
$$\mathbf{V}_{m+1} = 2\mathbf{V}_m - \mathbf{V}_{m-1}$$



Closed curve

- Vertices are of the form:

$$\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_m, \mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2$$



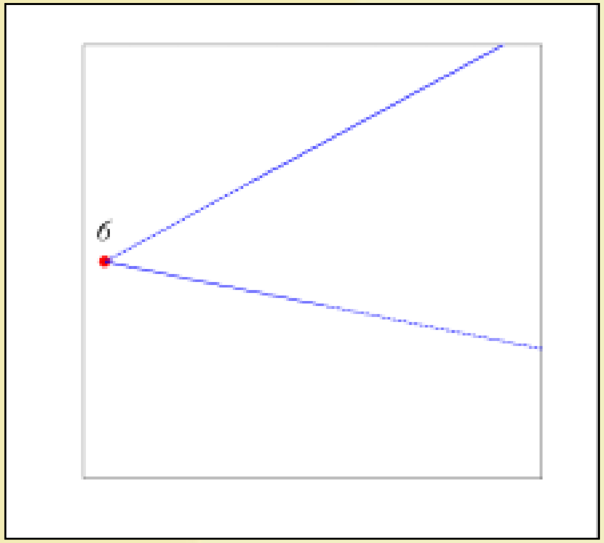
Double/triple interior vertex

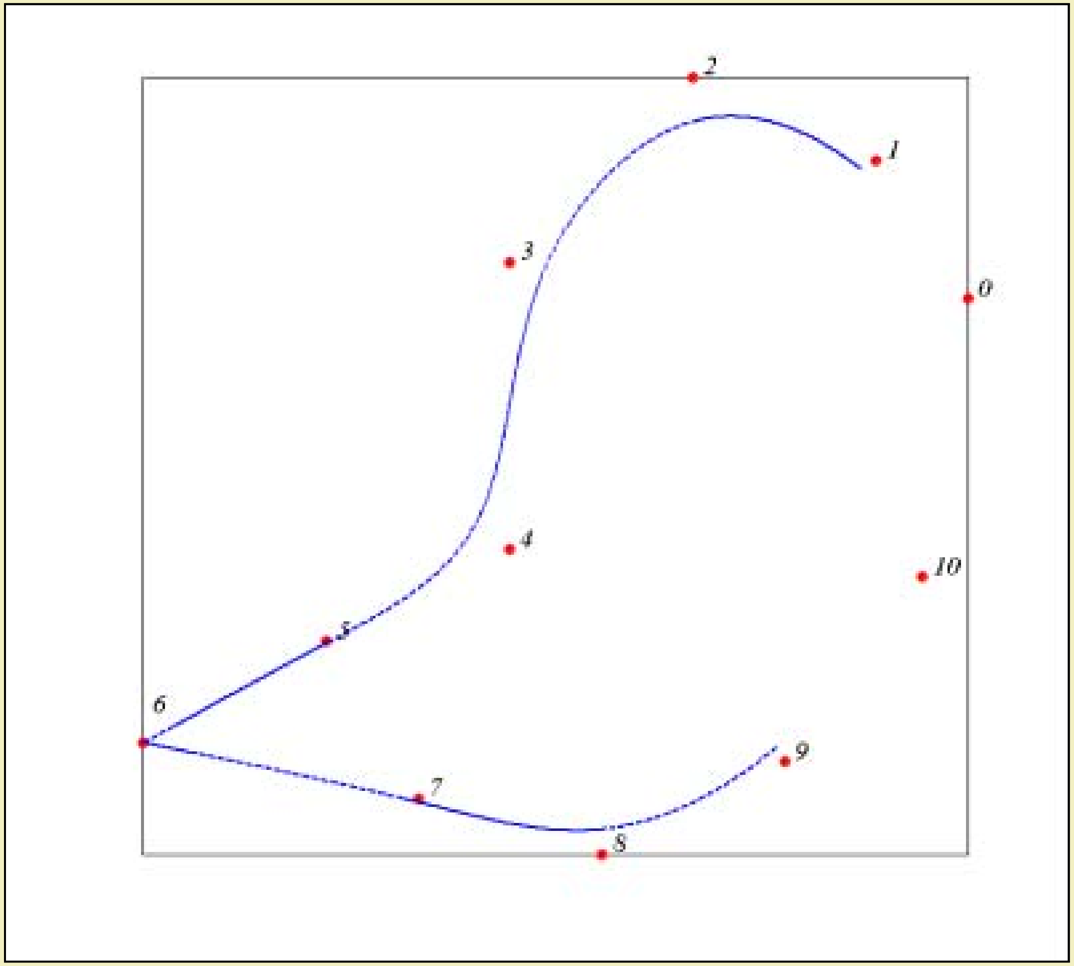
- Double vertex:

  $\mathbf{V}_6$

- Triple vertex:

  $\mathbf{V}_6$





Summary

- B-splines for geometric modeling
- Satisfies nice properties
- Manipulation by adding or changing vertices

Literature

- R.H. Bartels, J.C Beatty, B.A. Barksy:

  An Introduction to splines for use in Computer Graphics & Geometric Modeling