A story about Non Uniform Rational B-Splines

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Speakers

- 09-06: B-spline curves (W. Dijkstra)
- 16-06: NURBS (E. Shcherbakov)
- 30-06: B-spline surfaces (M. Patricio)

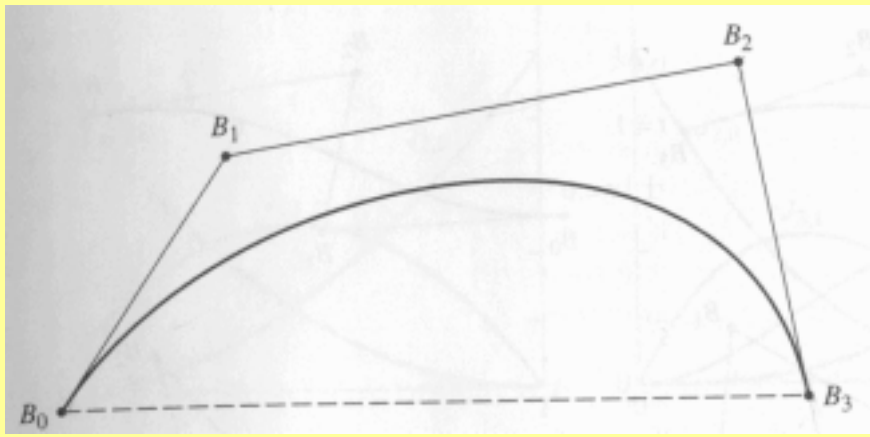
Outline

* B-spline
* Rational B-spline
* Conic sections

Classes of problems

* B-spline
* Rational B-spline
* Conic sections

- basic shape is arrived at by experimental evaluation or mathematical calculation → 'fitting' technique
- other design problems depend on *both* aesthetic and functional requirements (ab initio design)



B-Spline curve definition

* *B-spline*
* Rational B-spline
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$$
\begin{aligned}
P(t) &= \sum_{i=1}^{n+1} B_i N_{i,k}(t) \quad t_{min} \le t \le t_{max}, \quad 2 \le k \le n+1 \\
N_{i,1} &= 1 \text{ if } x_i \le t \le x_{i+1} \quad (0 \text{ otherwise}) \\
N_{i,k}(t) &= \frac{(t - x_i) N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t) N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}
\end{aligned}
$$

basis functions are defined by the Cox-de Boor recursion formulas

Properties of B-spline curves

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- sum of the B-spline basis functions for any parameter value is one
- each basis function is positive or zero
- precisely one maximum (except k=1)
- maximum order of the curve is one less of the number of control polygon vertices
- variation-diminishing properties (doest not oscillate)
- curve generally follows the shape of the control polygon
- curve is transformed by transforming the control polygon vertices
- curve lies within the convex hull of its control polygon

Convex hull

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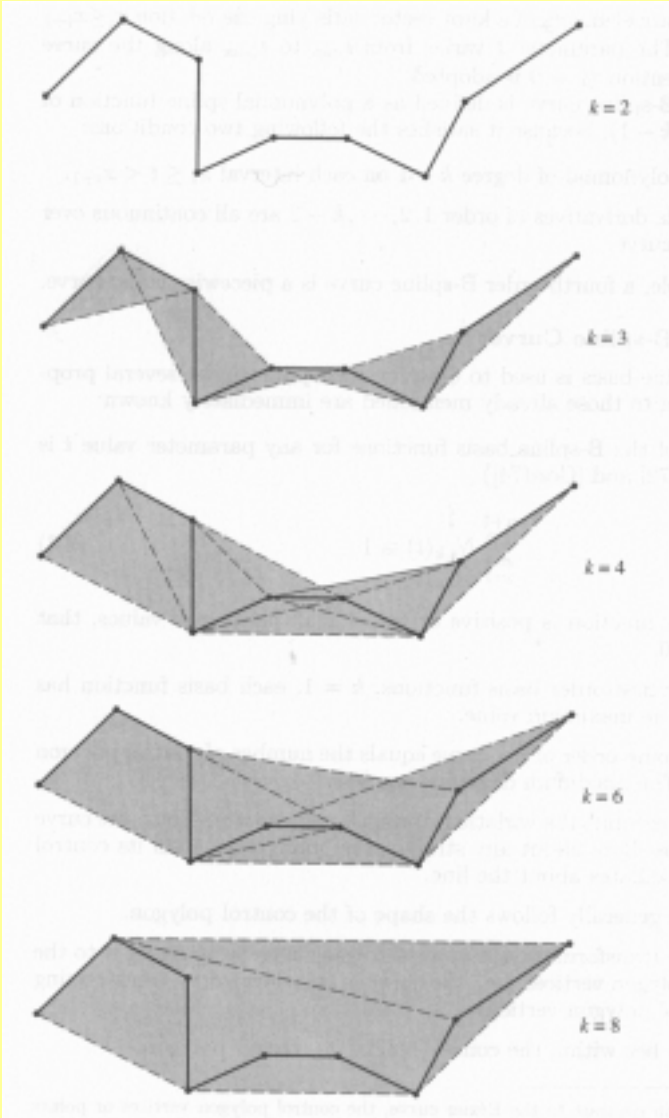

Knot Vectors

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The choice of a knot vector directly influences the resulting curve

The only requirements: monotonically increasing series of real numbers

Two types:

- periodic [0 1 2 3 4]
- open k =2 [0 0 1 2 3 4 4]

Two flavors:

- uniform
- nonuniform

Required number of knots = n + k +1

Recursion relation

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For a given basis function $N_{i,k}$ this dependence forms a triangular pattern

$$
\begin{array}{ccccccc}
N_{i,k} & & & & & & \\
N_{i,k-1} & N_{i+1,k-1} & & & & & \\
N_{i,k-2} & N_{i+1,k-2} & N_{i+2,k-2} & & & & \\
\vdots & & & \vdots & & & \\
N_{i,1} & N_{i+1,1} & N_{i+2,1} & N_{i+3,1} & \cdots & N_{i+k-1,1}
\end{array}
$$

Examples for different knot vectors

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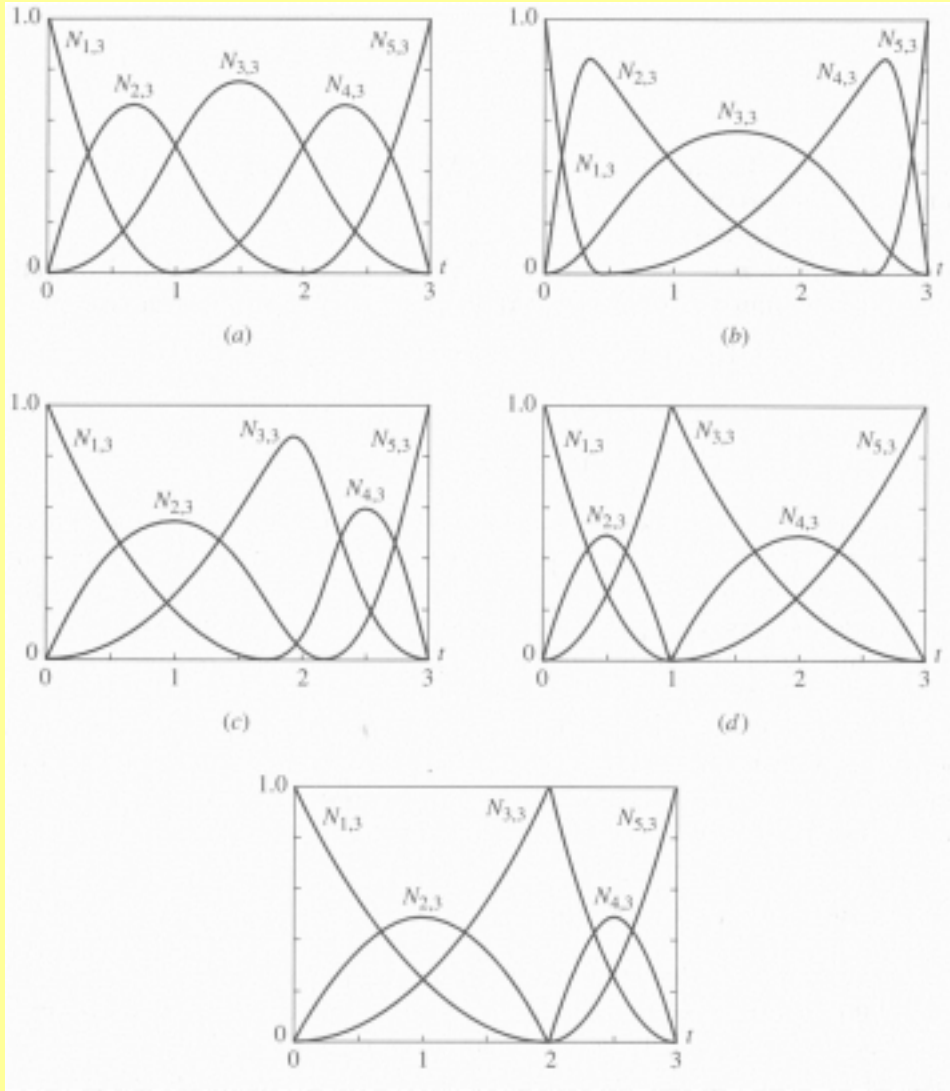

B-spline curve controls

* *B-spline*
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- changing the type of knot vector and hence basis function: periodic uniform, open uniform or nonuniform
- changing the order *k* of the basis function
- changing the number and position of the control polygon vertices
- using multiple polygon vertices
- using multiple knot values in the knot vector

Why further sharpening? (NURBS)

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Rational B-splines provide a single precise mathematical from capable of representing the common analytical shapes – lines, planes, conic curves including circles etc.

Interestingly enough, nonuniform rational B-splines have been an Initial Graphics Exchange Specification (IGES) standard since 1983 and incorporated into most of the current geometric modeling systems.

Rational B-spline curve definition

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Rational B-spline basis functions for weights h < 0 are also valid, but are not convenient. Algorithmically, convention 0/0=0 is adopted

$$P(t) = \sum_{i=1}^{n+1} B_i R_{i,k}(t)$$

$$R_{i,k} = \frac{h_i N_{i,k}(t)}{\sum_{i=1}^{n+1} h_i N_{i,k}(t)} \quad (h_i \geq 0)$$

Characteristics of NURBS

* B-spline
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Rational is generalization of nonrational; thus they carry forward all the analytic and geometric characteristics of their B-spline counterpars.

Also:

- a rational B-spline curve of order *k* is C^{k-2} continuous everywhere
- curve is invariant to any projective transformation (not only to affine)
- additional control capabilities due to weights

Conic sections

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Conic sections are described by quadratic equations, it is convenient to first consider a quadratic rational B-spline (k=3) defined by three polygon vertices (n+1=3) with knot vector [0 0 0 1 1 1]

$$P(t) = \frac{h_1 N_{1,3}(t) B_1 + h_2 N_{2,3}(t) B_2 + h_3 N_{3,3}(t) B_3}{h_1 N_{1,3}(t) + h_2 N_{2,3}(t) + h_3 N_{3,3}(t)}$$

Conic sections

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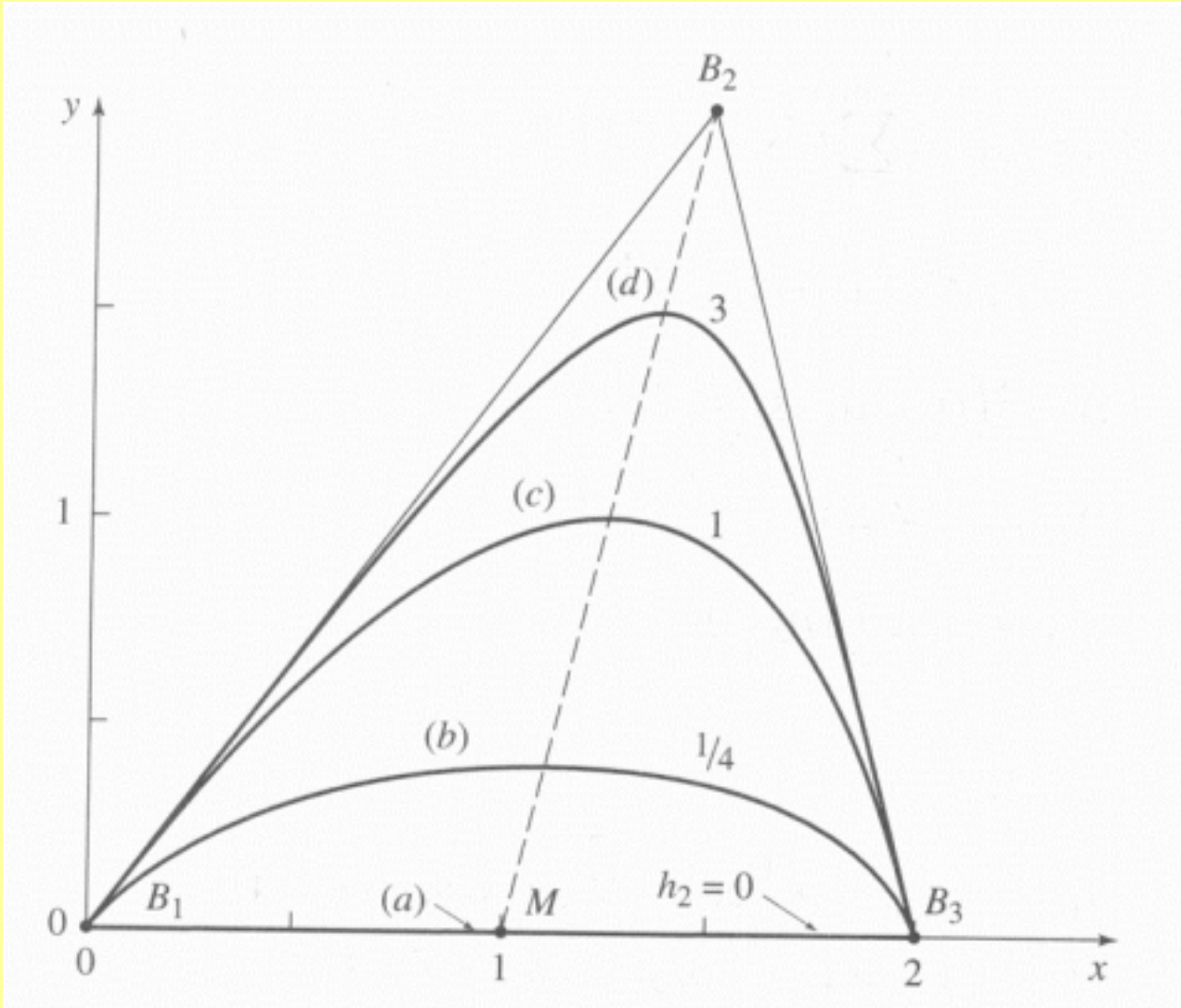

Conic sections

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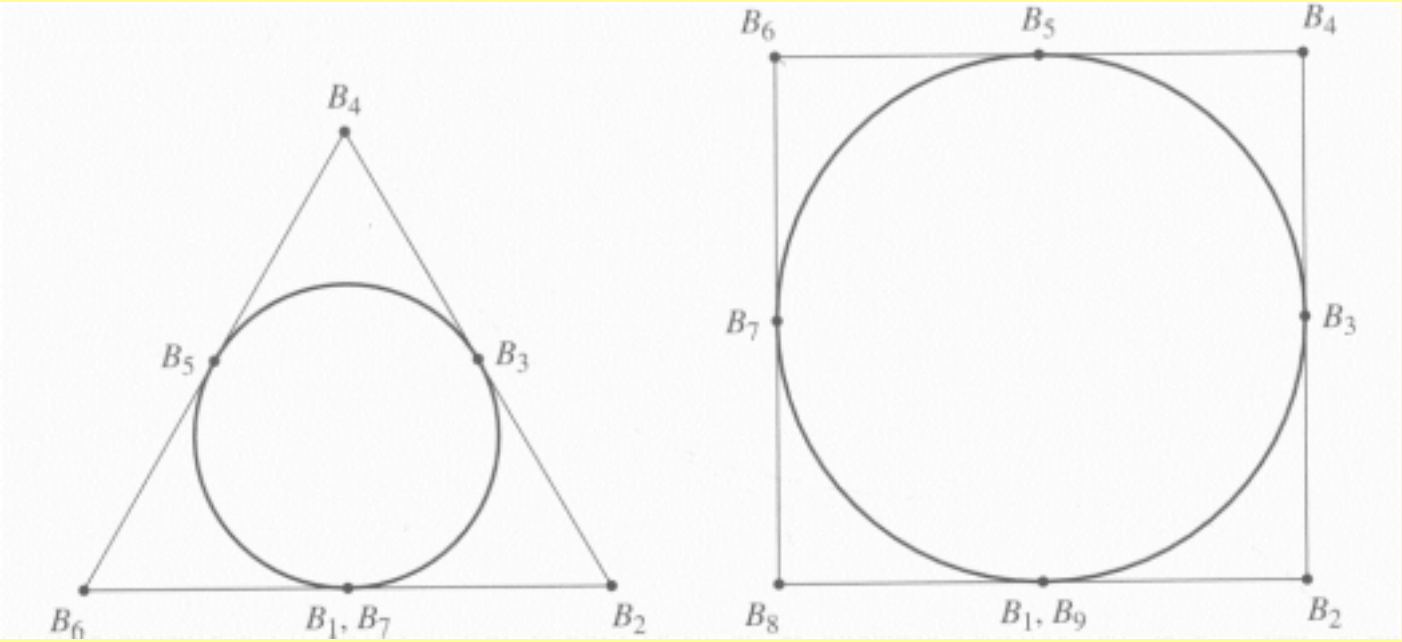
A full circle is formed by piecing together multiple segments



Conclusions

* Currently, NURBS curves are the standard for curve description in computer graphics
* Nice smooth properties
* Several ways to control the resulting curve provide great flexibility

References

Devid F. Rogers , *An introduction to NURBS,* Academic press 2001

R.H. Bartels, J.C Beatty, B.A. Barksy, *An Introduction to splines for use in Computer Graphics & Geometric Modeling*