

Eindhoven University of Technology

CASA seminar

Maximum Principle:
Maximum principle for elliptic equations

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Literature: M. E. Protter, H. F. Weinberger:

Maximum Principle in Differential Equations

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Motivation

- Qualitative properties of the solutions of PDE's (ODE's) inside the domain and on the boundaries
- Comparison principles
- Continuous dependence on data (boundary conditions, source term)
- Uniqueness for linear problems

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Overview

- The Laplace operator
- Second order elliptic operators
- Uniqueness theorems
- Comparison principle
- Nonlinear operators

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The Laplace operator

$u \in C^2(D)$, D domain in \mathbb{R}^n

Laplacian Δ is defined as

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

u is a **harmonic** function in D if

$$\Delta u = 0 \text{ in } D.$$

If u has a local maximum at $M \in D$. Then at this point

$$\frac{\partial u}{\partial x_1} = 0, \frac{\partial u}{\partial x_2} = 0, \cdots, \frac{\partial u}{\partial x_n} = 0$$

and

$$\frac{\partial^2 u}{\partial x_1^2} \leq 0, \frac{\partial^2 u}{\partial x_2^2} \leq 0, \cdots, \frac{\partial^2 u}{\partial x_n^2} \leq 0$$

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Therefore:

If

$$\Delta u > 0$$

at each point of D , u cannot attain maximum inside D .

More general:

If $b_1(x_1, x_2, \dots, x_n), b_2(x_1, x_2, \dots, x_n), \dots, b_n(x_1, x_2, \dots, x_n)$ are any bounded functions in D and if

$$\Delta u + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} > 0 \text{ in } D$$

then u cannot attain its maximum inside D .

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It is useful to establish maximum principle for differential inequalities which are not strict.

Restrict to $n = 2$ and thus

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

We write

$$\Delta u = \operatorname{div}(\operatorname{grad} u)$$

and use the **divergence theorem**:

If D is a bounded domain with smooth boundary ∂D , then

$$\iint_D \operatorname{div} w \, dx dy = \int_{\partial D} w \cdot n \, ds$$

n is outward normal unit vector (analog is valid in more dimensions).

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Therefore, for some disk K_r with radius r we have

$$\iint_{K_r} \Delta u \, dx dy = \iint_{K_r} \operatorname{div}(\operatorname{grad} u) \, dx dy = \int_{C_r} (\operatorname{grad} u) \cdot n \, ds = r \int_0^{2\pi} \frac{\partial u}{\partial r} \, d\theta.$$

Thus, if $\Delta u \geq 0$ in D , then

$$\int_0^{2\pi} \frac{\partial u}{\partial r} \, d\theta \geq 0.$$

Integration (in r) from 0 to R gives

$$u(x, y) \leq \frac{1}{2\pi R} \int_0^{2\pi} u(R, \theta) R \, d\theta = \frac{1}{2\pi R} \int_{C_R} u \, ds.$$

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If $\Delta u = 0$, applying the above procedure for u and $-u$ we obtain **Mean value theorem**:

Theorem. *If u is harmonic in D , then $u(x, y)$ is equal to its mean value taken over any circle in D with center at (x, y) :*

$$u(x, y) = \frac{1}{2\pi R} \int_{C_R} u \, ds.$$

Consequence is **Maximum principle**:

Theorem. *Let $\Delta u \geq 0$ in D . If u attains its maximum M at any point of D , then $u \equiv M$ in D .*

Definition. *A function u satisfying $\Delta u \geq 0$ in D is called **subharmonic**, while if $\Delta u \leq 0$ in D , u is called **superharmonic**.*

In \mathbb{R}^n , $\Delta u \geq 0$ implies $u(x_1, x_2, \dots, x_n) \leq \frac{1}{\omega_n R^{n-1}} \int_{C_R} u \, dS$ and above two theorems follow directly.

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Second order elliptic operators

Consider operators of the form

$$\mathcal{L} \equiv \sum_{i,j=1}^n a_{i,j}(x_1, x_2, \dots, x_n) \frac{\partial^2}{\partial x_i \partial x_j}.$$

We may assume that $a_{ij} = a_{ji}$.

Definition. The operator \mathcal{L} is called *elliptic* at $\mathbf{x} = (x_1, x_2, \dots, x_n)$ iff there is a positive quantity $\mu(\mathbf{x})$ such that

$$\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \mu(\mathbf{x}) \sum_{i=1}^n \xi_i^2$$

for all $(\xi_1, \xi_2, \dots, \xi_n)$. \mathcal{L} is *elliptic in domain D* if it is elliptic at each point of D . \mathcal{L} is *uniformly elliptic* in D if above inequality is satisfied in each point in D and if $(\exists \mu_0): (\forall \mathbf{x} \in D) \mu(\mathbf{x}) \geq \mu_0$.

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In other words: \mathcal{L} is elliptic if symmetric matrix $A = (a_{ij})$ is **positive definite**.

Consider linear transformation

$$\mathbf{y} = C\mathbf{x} \quad (*)$$

for $C = (c_{ij})$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$

Definition. Transformation $(*)$ is called a **rotation** or an **orthogonal transformation** iff C is orthogonal matrix, i.e. $C^T = C^{-1}$.

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Theorem. Suppose the operator

$$\mathcal{L} \equiv \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

is elliptic. Then under orthogonal transformation (*) the operator \mathcal{L} takes the form

$$\mathcal{L} \equiv \sum_{k,l=1}^n b_{kl} \frac{\partial^2}{\partial y_k \partial y_l} \quad (**)$$

where $b_{kl} = \sum_{i,j=1}^n a_{ij} c_{ki} c_{lj}$, $k, l = 1, 2, \dots, n$. Furthermore, the operator (**) is elliptic.

- Quantity $\mu(x)$ is unchanged
- Orthogonal transformations preserve uniform ellipticity
- If an operator with constant coefficients is elliptic at a point, it is uniformly elliptic in all n -space. In particular, Laplace operator is elliptic

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For any symmetric A there exists orthogonal C and diagonal D such that

$$D = CAC^{-1}.$$

Diagonal elements of D are called **eigenvalues** of A and rows of C **eigenvectors** of the corresponding linear transformation

If \mathcal{L} is elliptic all eigenvalues are **positive**. If \mathcal{L} is not elliptic, at least one eigenvalue is not positive.

Theorem. *A second order operator \mathcal{L} is elliptic at a point \bar{x} iff there is a linear transformation*

$$z_k = \sum_{j=1}^n d_{kj}x_j, \quad k = 1, 2, \dots, n,$$

such that at \bar{x} \mathcal{L} becomes the Laplacian in $\{z_k\}$ – coordinates.

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Definition. *The operator*

$$L + h \equiv \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + h$$

is said to be elliptic at \mathbf{x} (in D , uniformly elliptic) iff

$$\mathcal{L} \equiv \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

*is elliptic at \mathbf{x} (in D , uniformly elliptic). \mathcal{L} is called **principal part** of $L + h$.*

The maximum principle of E. Hopf

Consider

$$L[u] \equiv \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} > 0, \quad (*)$$

in D , where L is elliptic in D .

If u has a relative maximum at a point x we know that

$$\frac{\partial u}{\partial z_k} = 0, \quad \frac{\partial^2 u}{\partial z_k^2} \leq 0, \quad k = 1, 2, \dots, n$$

for any coordinates z_1, z_2, \dots, z_n obtained by linear transformation of x_1, x_2, \dots, x_n .

If L is elliptic, we can find a linear transformation of coordinates such that at x principal part becomes Laplacian.

Thus, if L is elliptic, a function u satisfying $(*)$ in D cannot have a maximum inside D

More general:

Theorem. *Let u satisfies the differential inequality*

$$L[u] \equiv \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} \geq 0$$

in a domain D where L is uniformly elliptic. Suppose a_{ij} and b_i are uniformly bounded. If u attains a maximum M at a point of D , then $u \equiv M$ in D .

- The assumptions of uniform ellipticity and boundedness of a_{ij} and b_{ij} can be weakened by boundedness of

$$\sum_{i=1}^n \frac{a_{ii}(x)}{\mu(x)} \quad \text{and} \quad \sum_{i=1}^n \frac{|b_i(x)|}{\mu(x)}$$

on any closed ball contained in the interior of D .

- Analogous minimum principle is valid for u satisfying $L[u] \leq 0$. Thus, if $L[u] = 0$ neither maximum nor minimum cannot be attained in D .

Theorem. *Let u satisfies the differential inequality*

$$(L + h)[u] \geq 0$$

with $h \leq 0$, L uniformly elliptic and the coefficients of L and h bounded. If u attains a nonnegative maximum M at and interior point of D , then $u \equiv M$.

- Weaker hypotheses:

$$\sum_{i=1}^n \frac{a_{ii}(x)}{\mu(x)}, \quad \sum_{i=1}^n \frac{|b_i(x)|}{\mu(x)} \quad \text{and} \quad \frac{-h(x)}{\mu(x)}$$

are bounded on any closed ball contained in D .

- Restriction $h \leq 0$ is essential.

Example. $u = e^{-r^2}$ is a solution of $\Delta u + (2n - 4r^2)u = 0$. But u has an absolute maximum at $r = 0$.

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Let $L[u] \geq 0$ in D . If u takes maximum it must be on ∂D

Suppose that $u \in C(D \cup \partial D)$ and that at $P \in \partial D$ u takes its maximum value. Then at that point the outward derivative

$$\frac{\partial u}{\partial \nu} \geq 0 \equiv \lim_{x \rightarrow P} (\nu \cdot \text{grad } u(x)) \geq 0$$

if this limit exists (here $\nu \cdot n > 0$). We can prove more:

Theorem. *Let u satisfies the differential inequality*

$$L[u] \equiv \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + hu \geq 0$$

in a domain D in which L is uniformly elliptic and $h \leq 0$. Suppose that $u \leq M$ in D and $u(P) = M$ for $P \in \partial D$. Assume that P lies on the boundary of a ball K_1 in D . If u is continuous in $D \cup P$ and if an outward directional derivative $\partial u / \partial \nu$ exists at P , then $\frac{\partial u}{\partial \nu} > 0$ at P , unless $u \equiv M$.

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- If $\partial u / \partial \nu$ does not exist at P , from the proof it still follows that

$$\liminf_{\alpha \rightarrow 0^+} \frac{u(x) - u(x - \alpha \nu)}{\alpha} > 0.$$

- Two examples for outward directional derivatives: the **normal derivative**, for $\nu = n$ and the **conormal derivative** for $\nu_i = \sum_{j=1}^n a_{ij} n_j$.

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Uniqueness theorems

$D \in \mathbb{R}^2$, **bounded** domain. Consider problem of finding $v \in C^2(D) \cup C(D \cup \partial D)$ satisfying Poisson equation and given Dirichlet boundary condition:

$$\Delta u \equiv \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = f(x, y) \text{ in } D$$
$$v = g \text{ on } \partial D.$$

This is called **first boundary value problem**.

This problem has at most one solution!

Boundedness of D important!

Example. $v = e^x \sin y$ on $D = (-\infty, \infty) \times (0, \pi)$. We need an additional condition at $x^2 + y^2 \rightarrow \infty$!

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$D \in \mathbb{R}^n$, bounded domain. Consider problem of finding $v \in C^2(D) \cup C(D \cup \partial D)$ satisfying in D

$$(L + h)[v] \equiv \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial v}{\partial x_i} + h(x)v = f$$

subject to

$$\frac{\partial v}{\partial \nu} + \alpha(x)v = g_1 \text{ on } \Gamma_1,$$

$$v = g \text{ on } \Gamma_2,$$

- For $\Gamma_2 = \emptyset$, $\partial/\partial \nu$ conormal derivative and
 - $\alpha \equiv 0$ we have the **Neumann problem** or **second B.V. problem**
 - $\alpha \neq 0$ we have the **third B.V. problem**
- For $\Gamma_1, \Gamma_2 \neq \emptyset$, we have a problem with **mixed boundary conditions**

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$D \in \mathbb{R}^n$, **bounded** domain. Consider problem of finding $v \in C^2(D) \cup C(D \cup \partial D)$ satisfying in D

$$(L + h)[v] \equiv \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial v}{\partial x_i} + h(x)v = f$$

subject to

$$\frac{\partial v}{\partial \nu} + \alpha(x)v = g_1 \text{ on } \Gamma_1,$$

$$v = g \text{ on } \Gamma_2,$$

$\Gamma_1 \cup \Gamma_2 = \partial D$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. Γ_1 and Γ_2 may be unconnected and even vacuous (empty).

Theorem. *Suppose v_1 and v_2 satisfy above equation and B.C.. Assume that each point on Γ_1 lies on the boundary of a ball in D . If L is uniformly elliptic, $h(x) \leq 0$ bounded and $\alpha(x) \geq 0$, then $v_1 \equiv v_2$, except when $h \equiv \alpha \equiv 0$ and $\Gamma_2 = \emptyset$, in which case $v_1 - v_2$ must be constant.*

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Example. Consider the steady state temperature distribution of a solid. Let $T(x, y, z)$ be the temperature distribution at the point $P(x, y, z)$ of a homogeneous, isotropic medium, occupying a bounded domain D . Suppose that part Γ_1 of ∂D is isolated and the temperature is prescribed on the remainder Γ_2 . In equilibrium state we have

$$\Delta T = 0 \text{ in } D$$

and

$$\frac{\partial T}{\partial n} = 0 \text{ on } \Gamma_1 \text{ and } T \text{ prescribed on } \Gamma_2.$$

- The equilibrium state temperature is unique
- T always lies between maximum and minimum on Γ_2
- An uneven temperature can be maintained only by supplying or removing heat on Γ_2 (flux is proportional to $\partial T / \partial n$)

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Comparison principle

Let u be a solution of

$$(L + h)[u] = f(x) \text{ in } D$$

satisfying

$$\begin{aligned} \frac{\partial u}{\partial \nu} + \alpha(x)v &= g_1(x) \text{ on } \Gamma_1, \\ v &= g_2 \text{ on } \Gamma_2(x). \end{aligned}$$

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Next we assume that z_1 and z_2 can be found such that

$$(L + h)[z_1] \leq f(x) \text{ in } D$$

$$\frac{\partial z_1}{\partial \nu} + \alpha(x)z_1 \geq g_1(x) \text{ on } \Gamma_1,$$

$$z_1 \geq g_2(x) \text{ on } \Gamma_2(x),$$

and

$$(L + h)[z_2] \geq f(x) \text{ in } D$$

$$\frac{\partial z_2}{\partial \nu} + \alpha(x)z_2 \leq g_1(x) \text{ on } \Gamma_1,$$

$$z_2 \leq g_2(x) \text{ on } \Gamma_2(x).$$

If the problem has solution for arbitrary continuous g_2 , then for a solution u it holds

$$z_2 \leq u \leq z_1 \text{ in } D.$$

Nonlinear operators

Let $F(x, y, u, p, q, r, s, t)$ be a continuously differentiable function of eight variables. We say that equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = f(x, y) \quad (*)$$

is **elliptic with respect to function u** if for all $(\zeta, \eta) \neq (0, 0)$ we have

$$F_r \zeta^2 + F_s \zeta \eta + F_t \eta^2 > 0.$$

Theorem. *Let u be a solution of $(*)$ in D and $u = g$ on ∂D . Let z and Z satisfy*

$$F(x, y, Z, Z_x, Z_y, Z_{xx}, Z_{xy}, Z_{yy}) \leq f(x, y) \leq F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) \quad \text{in } D$$

and $z \leq u \leq Z$ on ∂D . If for each $\theta \in [0, 1]$ F is elliptic w.r.t. $u + \theta(z - u)$ and $u + \theta(Z - u)$ in D and if $F_u \leq 0$ in D , then we have

$$z \leq u \leq Z \quad \text{in } D.$$