

Maximum Principles for Parabolic Equations

Kamyar Malakpoor

24 November 2004

Textbooks:

Friedman, A. Partial Differential Equations of Parabolic Type;
Protter, M. H, Weinberger, H. F, Maximum Principles in Differential Equations;

Outline

- ✚ Review of MP for the elliptic equations;
- ✚ MP for the heat equation $L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}$
- ✚ Weak MP for the parabolic equations;
 - Applications;
 - Comparison Principle;
 - Uniqueness Results;
- ✚ Strong MP for the parabolic equations;

Review of MP for the elliptic equations

➔ Consider the operator

$$Lu \equiv \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u \quad (\text{I})$$

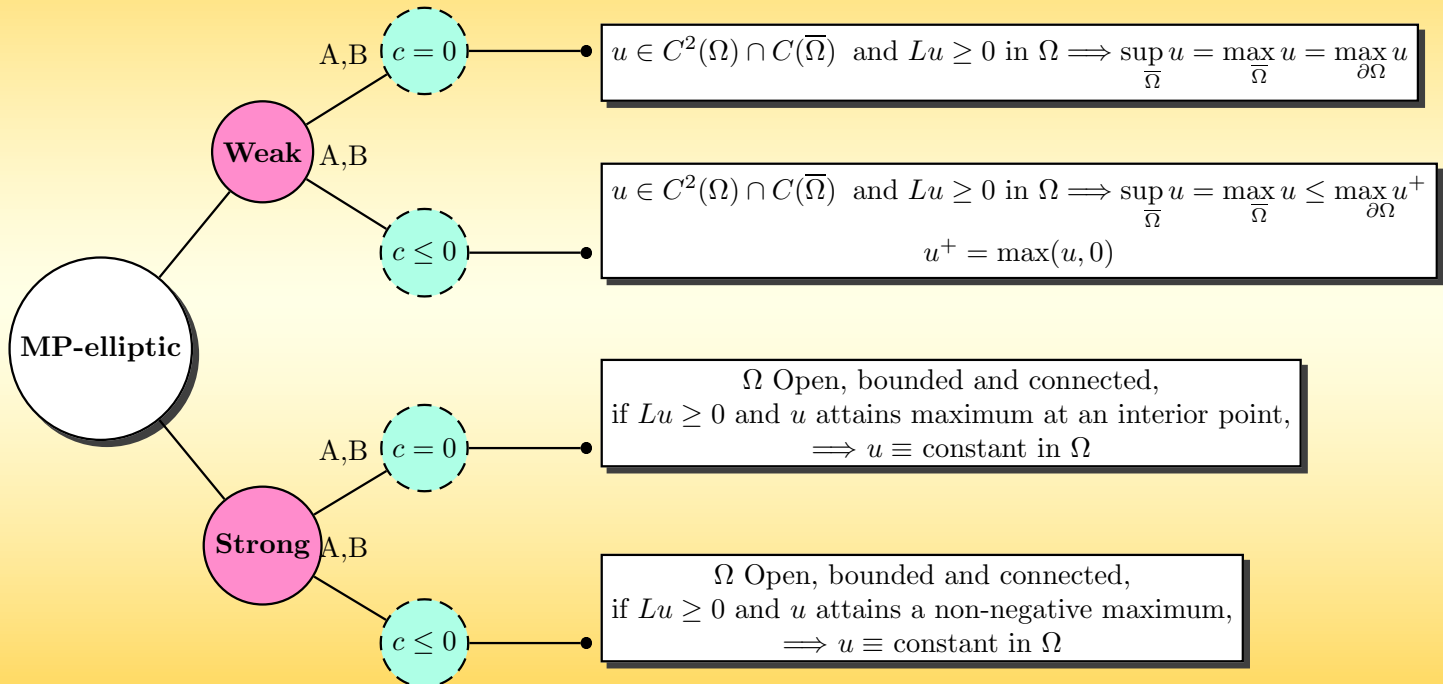
in an n -dimensional domain Ω (open and bounded).

➔ (A) We say that L is elliptic in Ω , if there exists $\lambda > 0$ such that for every $x \in \Omega$ and for any real vector $\xi \neq 0$,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j > \lambda |\xi|^2$$

➔ (B) We assume that the coefficients in L are bounded and continuous functions in D

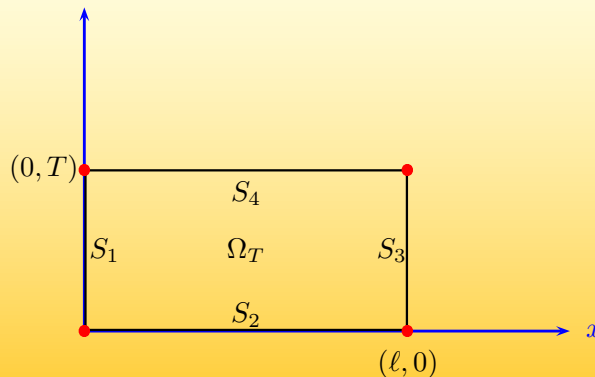
$$Lu \equiv \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$



MP for the Heat Equation $L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}$

✦ Suppose $u(x, t)$ satisfies the inequality $L(u) > 0$ in the rectangular region $\Omega_T = (0, \ell) \times (0, T]$ then u cannot have a (local) maximum at any interior point.

✦ For at such a point $\frac{\partial^2 u}{\partial x^2} \leq 0$ and $\frac{\partial u}{\partial t} = 0$, thereby violating $Lu > 0$



✦ Suppose $u(x, t)$ satisfies in $L(u) \geq 0$ in Ω_T . Then $\max_{\Omega_T} u = \max_{S_1 \cup S_2 \cup S_3} u$

- Define $M := \max_{S_1 \cup S_2 \cup S_3} u$. Let $(x_0, t_0) \in \Omega_T$, such that $M_1 := u(x_0, t_0) > M$.
- Define $v(x) := u(x) + \frac{M_1 - M}{2\ell^2}(x - x_0)^2$, then $v(x) < M_1$ on $S_1 \cup S_2 \cup S_3$ and $v(x_0, t_0) = M_1$,
- Furthermore $L(v) = L(u) + \frac{M_1 - M}{\ell^2} > 0$ on $\Omega_T \Rightarrow v$ cannot have an interior maximum.
- At a maximum on S_4 , $\partial^2 v / \partial x^2 \leq 0$ and therefore $\partial v / \partial t < 0$ and this contradicts with $u(x_0, t_0) = v(x_0, t_0) < M$.

Weak MP for the Parabolic Equations

✦ Consider the operator

$$Lu \equiv \underbrace{\sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u}_{Au} - \frac{\partial u}{\partial t} \quad (2)$$

in $\Omega_T = \Omega \times (0, T]$, with $T > 0$, and Ω domain in \mathbb{R}^n , (open and bounded).

✦ (A) We say that L is **parabolic** in Ω_T , if there exists $\lambda > 0$ such that for every $(x, t) \in \Omega_T$ and for any real vector $\xi \neq 0$,

$$\sum_{i,j=1}^n a_{ij}(x,t) \xi_i \xi_j > \lambda |\xi|^2$$

✦ (B) We assume that the coefficients in L are bounded functions in Ω_T

Weak MP for the Parabolic Equations(1)

$$\leftarrow Lu \equiv \underbrace{\sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u}_{Au} - \frac{\partial u}{\partial t}$$

\leftarrow Notation:

$$C^{(2,1)}(\Omega_T) = \left\{ u : \Omega_T \rightarrow \mathbb{R}; u, u_t, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in C(\Omega_T) \right\}$$

\leftarrow Define $\partial^* \Omega_T = \partial \Omega_T \setminus \Omega \times \{T\}$.

\leftarrow **Theorem:** Let (A),(B) hold and $c = 0$. If $u \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies $L(u) = A(u) - u_t \geq 0$, then

$$\sup_{\Omega_T} u = \max_{\Omega_T} u = \max_{\partial^* \Omega_T} u$$

✦ Proof.

- Suppose $L(u) > 0$ and max is attained at $(x_0, t_0) \in \Omega_T$. Therefore $\partial u / \partial x_i = \partial u / \partial t = 0$ at (x_0, t_0) and $D^2u := \left(\frac{\partial^2 u}{\partial x_i \partial x_j} (x_0, t_0) \right)_{i,j}$ is negative semi-definite, therefore

$$0 < L(u) = (a_{ij}) : D^2(u) \leq 0, \quad \text{contradiction!!}$$

- If the max is attained at (x_0, T) , then $\partial u / \partial t(x_0, T) \geq 0 \Rightarrow$

$$0 < L(u) = (a_{ij}) : D^2(u) - u_t \leq 0, \quad \text{contradiction!!}$$

- If $L(u) \geq 0$, then take $u^\epsilon = u - \epsilon t \Rightarrow$

$$L(u^\epsilon) = (A - \partial_t)(u - \epsilon t) = L(u) + \epsilon > 0$$

This implies that $\max_{\Omega_T} u^\epsilon = \max_{\partial^* \Omega_T} u^\epsilon$ for every $\epsilon > 0$.

The assertion follows as $\epsilon \searrow 0$.

Weak MP for the Parabolic Equations(2)

$$\leftarrow Lu \equiv \underbrace{\sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u}_{Au} - \frac{\partial u}{\partial t}$$

\leftarrow **Theorem:** Let (A),(B) hold and $c \leq 0$ implies that, if $u \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies $L(u) = A(u) - u_t \geq 0$, then

$$\sup_{\Omega_T} u = \max_{\overline{\Omega_T}} u \leq \max_{\partial^* \Omega_T} u^+$$

where $u = u^+ - u^-$, $u^+ = \max(u, 0)$.

✦ Proof.

- Suppose $L(u) > 0$, and that u has a nonnegative maximum at $(x_0, t_0) \in \Omega_T$, then

$$0 < L(u) = \underbrace{((a_{ij}) : D^2(u))}_{\leq 0} + \underbrace{c(x_0, t_0)}_{\leq 0} \underbrace{u}_{\geq 0} \leq 0, \quad \text{contradiction!!}$$

- If the max is attained at (x_0, T) , then $\partial u / \partial t(x_0, T) \geq 0 \Rightarrow$

$$0 < L(u) = \underbrace{(a_{ij}) : D^2(u)}_{\leq 0} \underbrace{-u_t}_{\leq 0} + \underbrace{c(x_0, T)}_{\leq 0} u \leq 0, \quad \text{contradiction!!}$$

✦ Proof.

- If $L(u) \geq 0$. Suppose $\Omega \subset \{\|x_1\| < d\}$. Consider $u_\epsilon = u + \epsilon e^{\alpha x_1} \Rightarrow$

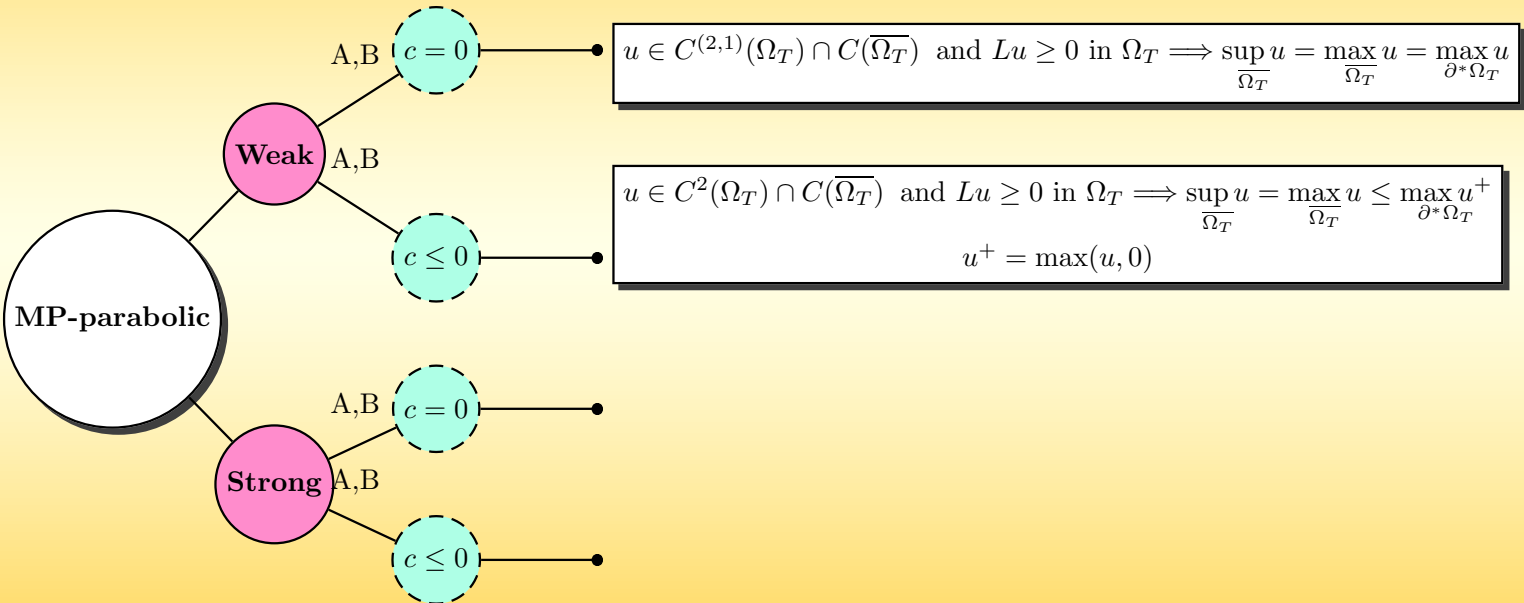
$$\begin{aligned} L(u_\epsilon) &= (A - \partial_t)(u + \epsilon e^{\alpha x_1}) \\ &= L(u) + \epsilon(\alpha^2 a_{11}(x, t) + \alpha b_1(x, t) + c(x, t))e^{\alpha x_1} \\ &\geq \epsilon(\alpha^2 \lambda - \alpha \|b_1\|_\infty - \|c\|_\infty)e^{\alpha x_1}. \end{aligned}$$
- By choosing α large enough, $L(u_\epsilon) > 0$, therefore

$$\sup_{\overline{\Omega_T}} u \leq \sup_{\overline{\Omega_T}} u_\epsilon \leq \max_{\overline{\Omega_T}} u_\epsilon^+ = \max_{\partial^* \Omega_T} u_\epsilon^+ \leq \max_{\partial^* \Omega_T} u^+ + \epsilon e^{\alpha d}$$

for every $\epsilon > 0$.

The assertion follows as $\epsilon \searrow 0$.

$$Lu \equiv \underbrace{\sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u}_{Au} - \frac{\partial u}{\partial t}$$



Applications

- ✦ In this section we derive bounds on solution u of the equation $L(u) = f$ in Ω_T .
- ✦ (1). Let (A) and (B) hold and $c(x, t) \leq 0$. If $L(u) = 0$ in Ω_T , then

$$\max_{\overline{\Omega_T}} |u| \leq \max_{\partial^* \Omega_T} |u|$$

(apply the weak MP to u and to $-u$).

- ✦ (2). Let (A) and (B) hold and $c(x, t) \leq \eta$. If $L(u) = 0$ in Ω_T , then

$$\max_{\overline{\Omega_T}} |u| \leq e^{\eta T} \max_{\partial^* \Omega_T} |u|$$

(apply (1) to $v := ue^{-\eta t}$. Indeed, $(A - \partial_t)(ue^{-\eta t}) = e^{-\eta t}(A(u) - \partial_t u + \eta u)$).

Applications (Continue)

- ✦ (3). Let (A) and (B) hold and $c(x, t) \leq 0$. Also assume that $\Omega \subset \{\|x_1\| < d\}$ and $a_{11}\lambda^2 + b_1\lambda \geq 1$ in Ω_T , for some positive constant λ . If $L(u) = f$ in Ω_T , then

$$\max_{\overline{\Omega_T}} |u| \leq \max_{\partial^*\Omega_T} |u| + (e^{\lambda d} - 1) \max_{\overline{\Omega_T}} |f|$$

define $w := \pm u - \max_{\partial^*\Omega} |u| - (1 - e^{\lambda x_1}) e^{\lambda d} \max_{\overline{\Omega_T}} |f|$, then $L(w) \geq 0$ in Ω_T , therefore $w \leq 0$ on $\partial^*\Omega_T$, and this results the above inequality.

Applications (Continue)

✦ (4). If in (3) the assumption $c(x, t) \leq 0$ replaced by $c(x, t) \leq \eta$, then

$$\max_{\overline{\Omega_T}} |u| \leq e^{\eta T} \left[\max_{\partial^* \Omega_T} |u| + (e^{\lambda d} - 1) \max_{\overline{\Omega_T}} |f| \right]$$

This follows by applying (3) to $v := ue^{\eta t}$.

Comparison Principle

✦ **Theorem.** Let (A) and (B) hold. Let $c \leq 0$ and suppose that $f(x, t, u)$ is a continuous function of variables x, t and u and satisfies the one-sided uniform Lipschitz condition in u

$$f(x, t, v) - f(x, t, u) \leq k(v - u), \quad \forall x, t, u, v, \quad v > u,$$

If $u, v \in C^{(2,1)}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy $Lu + f(x, t, u) \geq 0$ and $Lv + f(x, t, v) \leq 0$ in Ω_T , and $u \leq v$ in $\partial\Omega_T$, then

$$u \leq v, \quad \text{in } \Omega_T.$$

✦ **Proof.** $0 \leq L(u - v) + f(x, t, u) - f(x, t, v) \leq (L + k)(u - v)$, therefore

$$\max_{\overline{\Omega_T}}(u - v) \leq e^{(k + \|c\|_{\infty} T)} \max_{\partial^* \Omega_T}(u - v) \leq 0$$

Uniqueness Results

- ✦ The First initial boundary value problem consists of solving the differential equation

$$\begin{cases} Lu(x, t) = f(x, t), & \text{in } \Omega_T; \\ u(x, 0) = \varphi(x), & \text{on } \Omega \times \{0\}; \\ u(x, t) = g(x, t), & \text{on } \partial\Omega \times (0, T]. \end{cases}$$

- ✦ **Theorem.** Let (A) and (B) hold. Then there exists **at most one solution** to the above problem.

- ✦ **Proof.**

- The assumption (B) implies that $c(x, t)$ is bounded, $c(x, t) \leq \eta$. Define $v := ue^{\eta t}$. This transformation carries $Lu = 0$ into $\tilde{L}v := Lv - \eta v = 0$. Now the assertion of the theorem follows from the weak MP for v and $-v$.

Nonlinear Parabolic Equations

✦ Consider the nonlinear differential operator

$$Lu \equiv F\left(x, t, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\right) - \frac{\partial u}{\partial t},$$

where F is a nonlinear function of its arguments.

✦ We say that F is parabolic at a point (x_0, t_0) if for any $p, p_1, \dots, p_n, p_{11}, \dots, p_{nn}$, the matrix

$$\left(\frac{\partial F(x_0, t_0, p, p_i, p_{ii})}{\partial p_{hk}} \right)$$

is positive definite.

✦ If $Lu^1 = Lu^2$ in the domain Ω_T then, by the mean value theorem,

$$\begin{aligned}\frac{\partial(u^1 - u^2)}{\partial t} &= F(x, t, u^1, \frac{\partial u^1}{\partial x_i}, \frac{\partial^2 u^1}{\partial x_i \partial x_j}) - F(x, t, u^2, \frac{\partial u^2}{\partial x_i}, \frac{\partial^2 u^2}{\partial x_i \partial x_j}) \\ &= \sum a_{hk} \frac{\partial^2(u^1 - u^2)}{\partial x_h \partial x_k} + \sum b_h \frac{\partial(u^1 - u^2)}{\partial x_h} + c(u^1 - u^2),\end{aligned}$$

where a_{hk}, b_h, c are continuous functions provided $\partial F/\partial p, \partial F/\partial p_h, \partial F/\partial p_h k$ are continuous functions.

- ✦ (a_{hk}) is positive definite matrix.
- ✦ Applying the previous theorem, we conclude that there exists at most one solution to $Lu = 0$.

Strong MP for the Parabolic Equations

- ✦ Theorem. Let Ω be open, bounded, and connected in \mathbb{R}^n . Let (A) and (B) hold. Let $u \in C^{(1,2)}(\Omega_T) \cap C(\overline{\Omega_T})$ with $Lu = Au - \partial_t u \geq 0$, then
- If $c \equiv 0$, then u cannot have a global maximum in Ω_T , unless u is constant.
 - If $c \leq 0$, then u cannot have a global nonnegative maximum in Ω_T , unless u is constant.

$$Lu \equiv \underbrace{\sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u}_{Au} - \frac{\partial u}{\partial t}$$

