Homogenisation Theory

in the

Finite Elements Method

Nico van der Aa

March 23th 2005

Overview of previous talks

- Yves van Gennip (26-01-2005)
  An introduction to homogenisation
- Miguel Patricio (09-02-2005)
  Homogenisation for Advection-Diffusion Equations
- Heike Gramberg (23-02-2005)
  Homogenization for the Stokes problem
- Matthias Röger (09-03-2005)
  Homogenisation theory for partial differential equations

Upcoming talk

- Mark van Kraaij (13-04-2005)
  Homogenisation theory in Finite Volume Method

Outline of my talk

Goal

My goal is to show an idea of how homogenisation theory can be applied to the Finite Element Method.

Way

By taking a specific test problem, I hope the idea becomes visible.

Sources

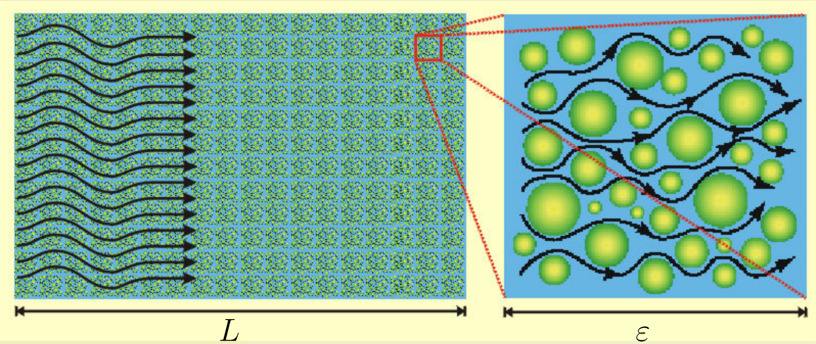
A Multiscale Finite Element Method for Elliptic Problems in Composite Materials and Porous Media
Thomas Y. Hou and Xiao-Hui Wu
Journal of Computational Physics **134**, 169-189 (1997)

Convergence of a Multiscale Finite Element Method for Elliptic Problems with Rapidly Oscillating Coefficients
Thomas Y. Hou and Xiao-Hui Wu
Mathematics of Computation **68**(227), 913-943 (1999)

Short recapitulation

Setting

A physical problem has a two-scale solution



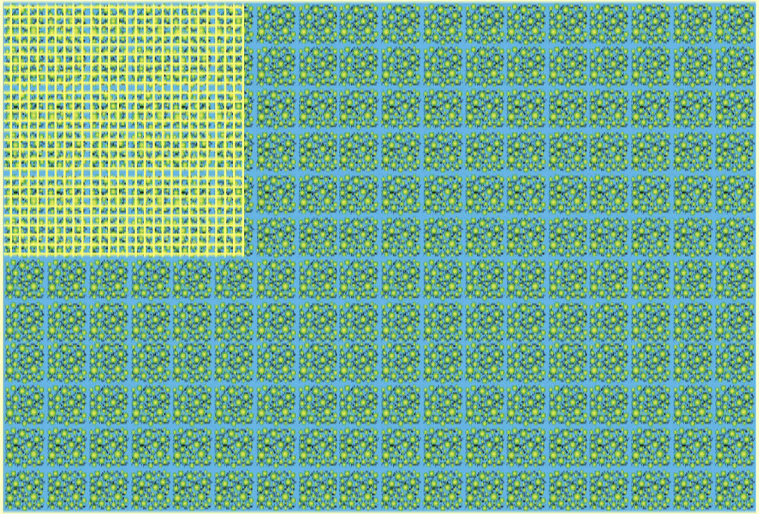
Problem

The major difficulty of direct solutions is the scale of computation

Goal

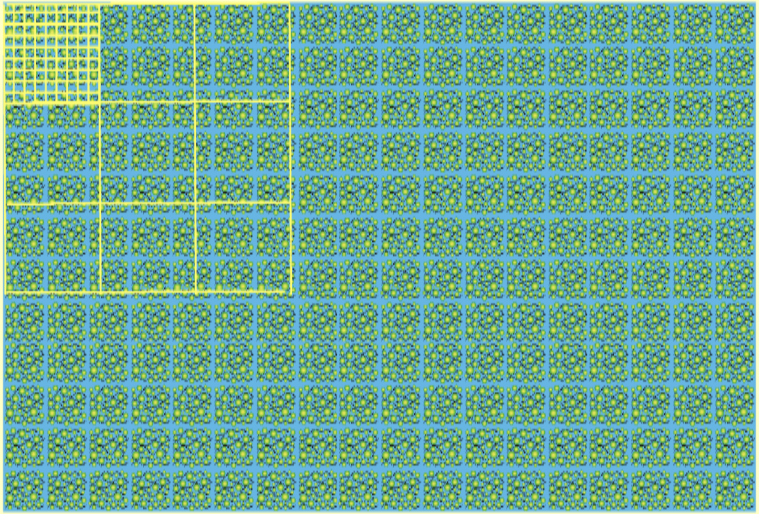
Obtain the large scale solution accurately and efficiently without resolving the small scale details

Motivation



One solution can be obtain by taking a very fine mesh (smaller than the small scale ε) and solve it with the ordinary Finite Element Method. This method requires a lot of memory.

Motivation



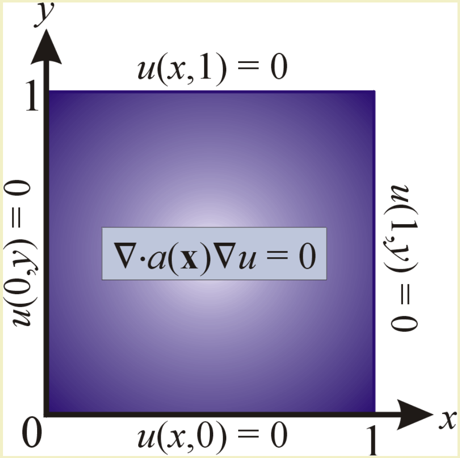
Another solution method is to use a coarse mesh (larger than the small scale ε) and capture the small scale inside the basis functions.
Note that the number of elements remains the same!

Problem statement

Equation

$$\nabla \cdot \boldsymbol{a}(\boldsymbol{x}/\varepsilon)\nabla u = f \quad \text{in } \Omega = (0,1) \times (0,1)$$

- symmetric
- positive definite
- bounded
- periodic



Boundary condition

$$u = 0 \quad \text{on } \partial\Omega$$

Context in porous flows

The equation is the pressure equation for a single phase steady flow through a porous medium and the conductivity tensor $\boldsymbol{a}$ is the ratio of the permeability tensor κ and the fluid viscosity μ, and u represents the pressure.

Finite Element formulation

- Variational problem

  Multiply original function with a test function w and integrate over the whole domain Ω

  $$\int_\Omega w \cdot (-\nabla \cdot a\nabla u)dV = \int_\Omega wf\ dV$$

  Use the vector identity $\nabla \cdot (u\boldsymbol{V}) = u\nabla \cdot \boldsymbol{V} + \boldsymbol{V} \cdot \nabla u$ and Gauss' divergence theorem

  $$-\int_{\partial\Omega} w \cdot (a\nabla u) \cdot \boldsymbol{n}\ dS + \int_\Omega \nabla w \cdot a\nabla u\ dV = \int_\Omega wf\ dV$$

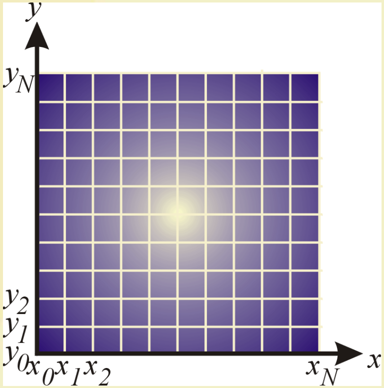
  Use boundary condition $u = 0$ at $\partial\Omega$

  $$\int_\Omega \nabla w \cdot a\nabla u\ dV = \int_\Omega wf\ dV$$

- Construction of mesh
- Construction of base functions
- Computation of the stiffness matrix
- Compute the solution of the discrete system

Finite Element formulation

- Variational problem
- Construction of mesh
  Let $0 < h \leq 1$ and $\mathcal{K}^h$ be a partition of Ω by a collection of rectangles K.



- Construction of base functions
- Computation of the stiffness matrix
- Compute the solution of the discrete system

Finite Element formulation

- Variational problem
- Construction of mesh
- Construction of base functions

  Define for each element $K \in \mathcal{K}^h$ a set of nodal basis $\{\phi_K^i, i = 1, ..., 4\}$ and assume that the base functions are continuous across the boundaries of the elements.

  More in detail later.
- Computation of the stiffness matrix
- Compute the solution of the discrete system

Finite Element formulation

- Variational problem
- Construction of mesh
- Construction of base functions
- Computation of the stiffness matrix

Use $u = \sum_{j=1}^{N} u_j \phi_j$ and $w = \phi_i$ to obtain:

$$\sum_{j=1}^{N} \underbrace{\left(\int_{\Omega} \nabla\phi_i \cdot a \nabla\phi_j \, dV \right)}_{A_{ij}} u_j = \underbrace{\int_{\partial\Omega} \phi_i f \, dV}_{b_i}$$

This results in a linear system of N equations with N unknowns

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$$

- Compute the solution of the discrete system

Finite Element formulation

- Variational problem
- Construction of mesh
- Construction of base functions
- Computation of the stiffness matrix
- Compute the solution of the discrete system

  Solution of the linear system:

  $$\boldsymbol{u} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

  The solution is than given by

  $$u = \sum_{j=1}^{N} u_j \phi_j$$

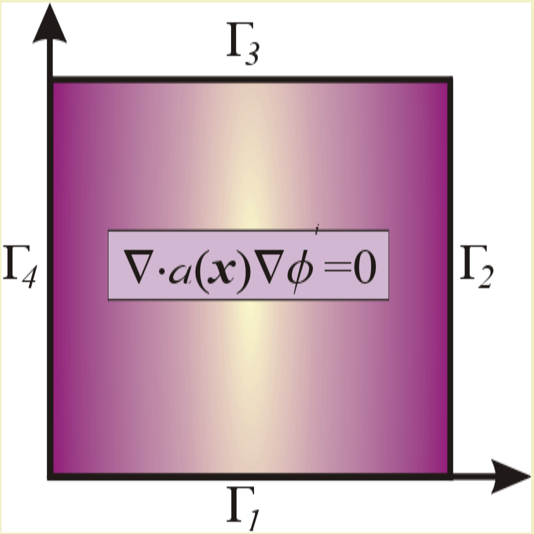
Boundary condition for the base function

The basis functions ϕ^i have to satisfy the following

- $\nabla \cdot a(\boldsymbol{x})\nabla\phi^i = 0$ in $K \in \mathcal{K}^h$;
- $\phi^i(\boldsymbol{x}_j) = \delta_{ij}$ for all nodal points $\boldsymbol{x}_j$ of K.

Choices for the boundary conditions for ϕ^i:

- linearly varying boundary condition;
- oscillatory boundary condition;
- oversampling method.



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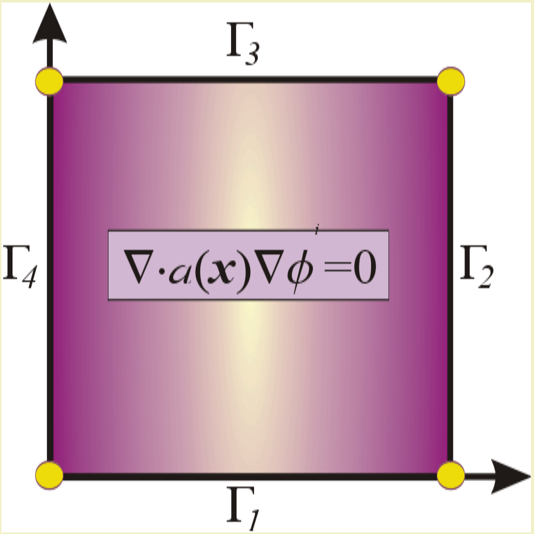

Boundary condition for the base function

The basis functions ϕ^i have to satisfy the following

- $\nabla \cdot a(\boldsymbol{x}) \nabla \phi^i = 0$ in $K \in \mathcal{K}^h$;
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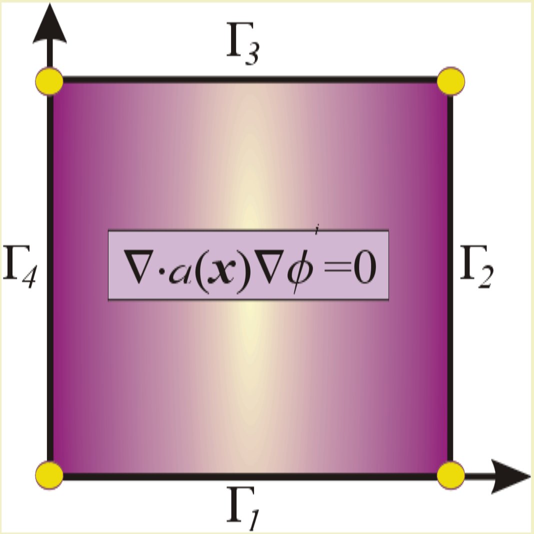

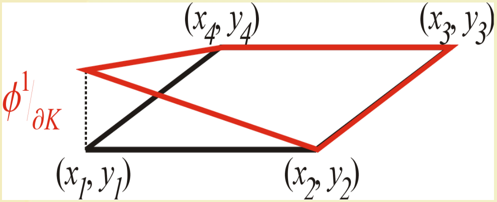

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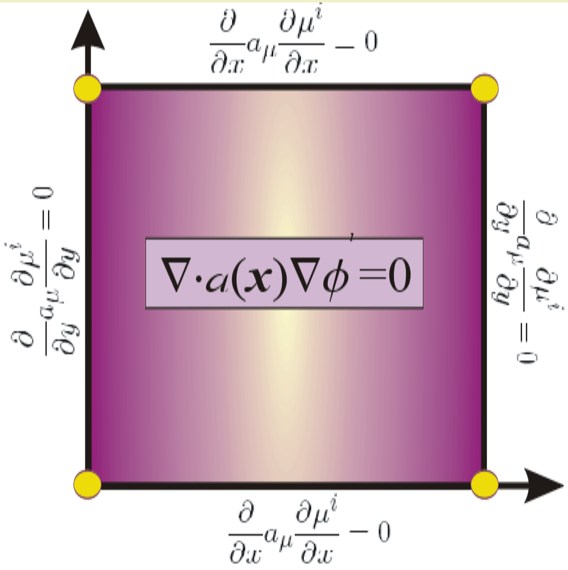
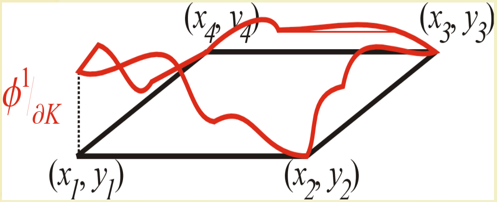
- $\phi^i(\boldsymbol{x}_j) = \delta_{ij}$ for all nodal points $\boldsymbol{x}_j$ of K.

Choices for the boundary conditions for ϕ^i:

- linearly varying boundary condition;
- oscillatory boundary condition:

  Delete the terms with partial derivatives in the direction normal to ∂K of the original equation.
- oversampling method.





Boundary condition for the base function

The basis functions ϕ^i have to satisfy the following

- $\nabla \cdot a(\boldsymbol{x})\nabla\phi^i = 0$ in $K \in \mathcal{K}^h$;
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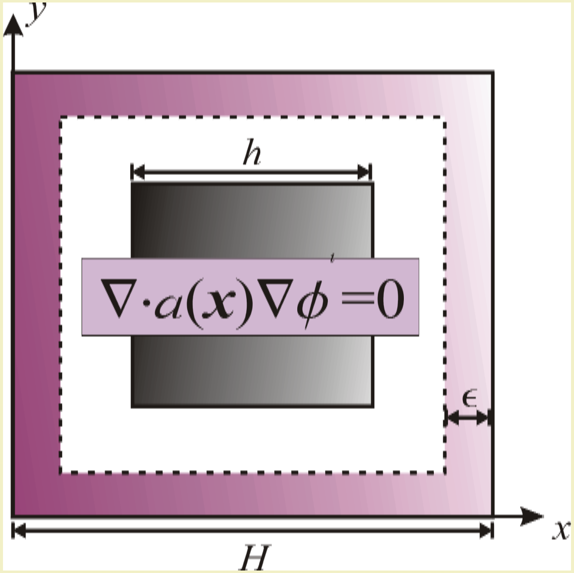
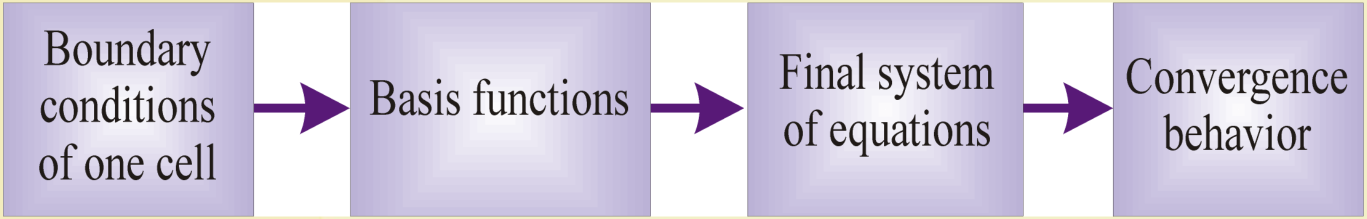
Choices for the boundary conditions for ϕ^i:

- linearly varying boundary condition;
- oscillatory boundary condition;
- oversampling method:

  Overcome the influence of the boundary layer, which is thin ($\mathcal{O}(\varepsilon)$), by sampling in a domain with size larger than $h+\varepsilon$ and use only the interior information to construct the base functions.



Remark on boundary conditions



For example, take a look at two results for two different problems

TABLE 1. Results of Example 7.1 ($h < \epsilon$, $M = 8$, $u_{max} \approx 0.02$).

ϵ	N	MFEM-O		MFEM-L		LFEM	
		$\|E\|_{l^2}$	rate	$\|E\|_{l^2}$	rate	$\|E\|_{l^2}$	rate
0.08	64	5.60e-4		6.90e-5		2.55e-4	
	128	2.32e-4	1.3	1.58e-5	2.1	6.65e-5	1.9
	256	7.11e-5	1.7	3.58e-6	2.1	1.66e-5	2.0
	512	1.87e-5	1.9	7.09e-7	2.3	3.92e-6	2.1
0.04	128	5.82e-4		5.65e-5		2.39e-4	
	256	2.39e-4	1.3	1.23e-5	2.2	6.20e-5	1.9
	512	7.33e-5	1.7	2.71e-6	2.2	1.55e-5	2.0
0.02	256	5.92e-4		5.10e-5		2.32e-4	
	512	2.42e-4	1.3	1.08e-5	2.2	5.98e-5	2.0
	1024	7.42e-5	1.7	2.32e-6	2.2	1.50e-5	2.0

General problem

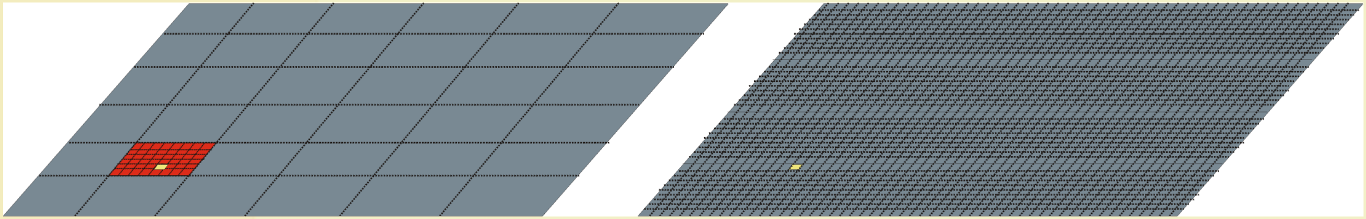
TABLE 4. Results of Example 7.2 ($h < \epsilon$, $M = 8$, $u_{max} \approx 0.87$).

ϵ	N	MFEM-O		MFEM-L		LFEM	
		$\|E\|_{l^2}$	rate	$\|E\|_{l^2}$	rate	$\|E\|_{l^2}$	rate
0.08	64	3.62e-5		3.73e-4		6.90e-4	
	128	8.94e-6	2.0	9.74e-5	1.9	1.79e-4	1.9
	256	2.24e-6	2.0	2.43e-5	2.0	4.48e-5	2.0
0.04	128	1.67e-5		3.88e-4		7.14e-4	
	256	4.11e-6	2.0	1.01e-4	1.9	1.85e-4	1.9
	512	8.62e-7	2.3	2.52e-5	4.0	4.62e-5	2.0
0.02	256	1.11e-5		3.85e-4		7.12e-4	
	512	2.74e-6	2.0	1.00e-4	1.9	1.84e-4	1.9
	1024	3.97e-7	2.8	2.49e-5	2.0	4.60e-5	2.0

Nice problem

Implementation

The implementation of the two-scale method implies mapping of basis functions inside one element of the fine mesh onto one element of the coarse mesh and another mapping of the coarse mesh onto the whole domain.



Two-scale method   Ordinary FEM

It would have taken too much time to implement it for this seminar.

Complexity

- Ordinary FEM:

  Complexity of Memory Usage $\Rightarrow \mathcal{O}(M^2 \times N^2)$
  Complexity of Operation Count $\Rightarrow \mathcal{O}(M^2 \times N^2)$

- Multiscale FEM:

  Complexity of Memory Usage $\Rightarrow \mathcal{O}(M^2 + N^2)$
  Complexity of Operation Count $\Rightarrow \mathcal{O}(N^2 + (d-1)(N^2 \times M^2))$

Remark: if $h = n\varepsilon$ with $n \in \mathbb{N}$, then the basis functions of one cell can be used for all other cells, since the governing differential equation and its boundary conditions are identical for all cells.
Complexity of Operation Count $\Rightarrow \mathcal{O}(N^2 + (d-1)(N^2 + M^2))$.

Basic Features of Homogenisation Theory

Asymptotic Expansion

$$u(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}) = \boxed{u_0(\boldsymbol{x})} + \varepsilon u_1(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}) - \varepsilon\theta_\varepsilon + \mathcal{O}(\varepsilon^2)$$

↳ Large scale solution

Homogenized equation

$$\nabla \cdot a^*(\boldsymbol{x}/\varepsilon)\nabla u_0 = f \text{ in } \Omega, \quad u_0 = 0 \text{ on } \partial\Omega$$

Correction term

Since $u_1 \neq 0$ on $\partial\Omega$, the boundary condition $u|_{\partial\Omega}$ is enforced through θ_ε, which is given by

$$\nabla \cdot a(\boldsymbol{x}/\varepsilon)\nabla\theta_\varepsilon = 0 \text{ in } \Omega, \quad \theta_\varepsilon = u_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \text{ on } \partial\Omega$$

Remark

This theory is only used for analysis purposes.

Mesh size

- Case 1: $h < \varepsilon$.
  - Error $(H^1(\Omega)) \sim \mathcal{O}(h/\varepsilon)$.
  - Multiscale method is similar to the ordinary FEM.
- Case 2: $h > \varepsilon$.
  - Error $(H^1(\Omega)) \sim \mathcal{O}(\varepsilon/h)$.
  - Ordinary FEM fails.

TABLE I

Results for $\varepsilon = 0.005$

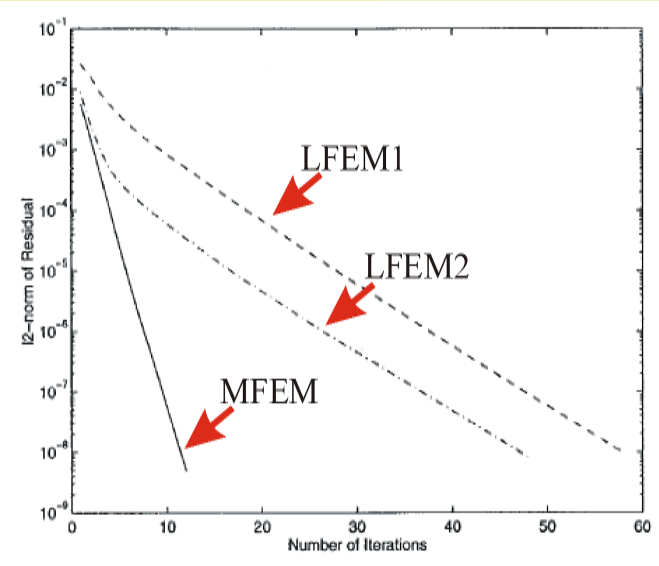
Mesh		MFEM-O			MFEM-L		
N	M	$\|E\|_\infty$	$\|E\|_{l^2}$	rate	$\|E\|_\infty$	$\|E\|_{l^2}$	rate
32	64	4.89e-5	2.52e-5		1.79e-4	9.73e-5	
64	32	1.06e-4	5.79e-5	−1.20	3.86e-4	2.13e-4	−1.13
128	16	1.74e-4	9.65e-5	−0.74	7.32e-4	4.10e-4	−0.94
256	8	3.76e-4	2.10e-4	−1.12	1.40e-3	7.83e-4	−0.93
512	4	1.77e-4	9.88e-5	1.09	1.00e-3	5.61e-4	0.48

TABLE III

Results for the Oversampling Method ($\varepsilon = 0.005$)

Mesh		$M_S = 128$		$M_S = 256$	
N	M	$\|E\|_\infty$	$\|E\|_{l^2}$	$\|E\|_\infty$	$\|E\|_{l^2}$
32	64	3.08e-5	1.53e-5	3.59e-5	8.14e-6
64	32	4.99e-5	2.06e-5	3.32e-5	1.14e-5
128	16	4.65e-5	1.51e-5	4.42e-5	8.07e-6
256	8	3.66e-5	1.63e-5	2.53e-5	7.26e-6
512	4	1.64e-5	3.42e-6	1.63e-5	5.04e-6

Strange result



Method	total number of elements	Size of smallest element
MFEM	$256 \times 32 = 8192$	1.22×10^{-4}
LFEM1	256	3.91×10^{-3}
LFEM2	512	1.95×10^{-3}

But $\varepsilon = \sqrt{2}/1000 \approx 1.41 \times 10^{-3}$, so we compare apples and oranges.

Resonance effect

Previously shown:

- if $h < \varepsilon$ then the error is $\mathcal{O}(\varepsilon/h)$.
- if $h > \varepsilon$ then the error is $\mathcal{O}(h/\varepsilon)$.

When $h \approx \varepsilon$ a so-called resonance effect occurs, but what is the physical reason for this effect?

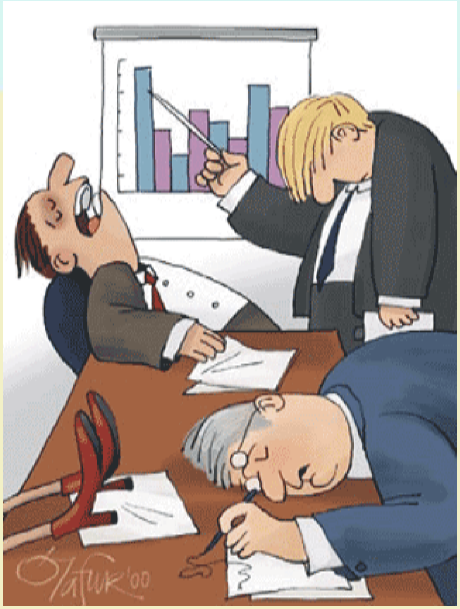
Results for $\varepsilon = 0.005$

Mesh		MFEM-O			MFEM-L		
N	M	$\|E\|_\infty$	$\|E\|_{l^2}$	rate	$\|E\|_\infty$	$\|E\|_{l^2}$	rate
32	64	4.89e-5	2.52e-5		1.79e-4	9.73e-5	
64	32	1.06e-4	5.79e-5	−1.20	3.86e-4	2.13e-4	−1.13
128	16	1.74e-4	9.65e-5	−0.74	7.32e-4	4.10e-4	−0.94
256	8	3.76e-4	2.10e-4	−1.12	1.40e-3	7.83e-4	−0.93
512	4	1.77e-4	9.88e-5	1.09	1.00e-3	5.61e-4	0.48

Results for $\varepsilon/h = 0.64$ and $M = 16$

		MFEM-O		MFEM-L		LFEM	
N	ε	$\|E\|_{l^2}$	rate	$\|E\|_{l^2}$	rate	$M\,N$	$\|E\|_{l^2}$
16	0.04	6.23e-5		3.54e-4		256	1.34e-4
32	0.02	8.43e-5	−0.44	3.90e-4	−0.14	512	1.34e-4
64	0.01	9.32e-5	−0.14	4.04e-4	−0.05	1024	1.34e-4
128	0.005	9.65e-5	−0.05	4.10e-4	−0.02	2048	1.34e-4

Conclusions



- FEM can be adjusted to two-scale problems;
- Homogenisation theory can help in the analysis;
- ...

Special thanks to Pavel Kagan for the inspiring discussions.

Next seminar is on April 13th given by Mark van Kraaij.

Questions ?