Finite Volume methods for steady problems

Numerical Solution of Convection-Diffusion problems

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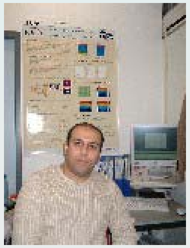
Seminar

http://www.win.tue.nl/casa/meetings/seminar/index.html

- May 18: Numerical solution of convection-diffusion problems, an introduction

- May 25: Numerical solution of convection-diffusion problems: difference schemes for steady problems

June 29



July 6

Outline

- Main idea of finite volume
  - conservation
  - different approaches
- Numerical discretization
  - quadrature rules
  - interpolation schemes
- Different grid types

The main idea

Convection-diffusion equation

$$\rho\,\mathbf{v}\cdot\nabla\Phi = \nabla\cdot(\Gamma\nabla\Phi) + s \quad \text{in}\,\Omega$$

- $\rho\mathbf{v}$ field is divergence free
- Apply Gauss' theorem

$$\int_S \rho\Phi\,\mathbf{v}\cdot\mathbf{n}\,dS = \int_S \Gamma\nabla\Phi\cdot\mathbf{n}\,dS + \int_V s\,dV \quad \text{for any } V \subseteq \Omega$$

- Want a numerical method that satisfies a discrete equivalent of

Example (time-dep. problem)

- Φ is the concentration of a passive tracer transported by a velocity field **v** in a closed domain.

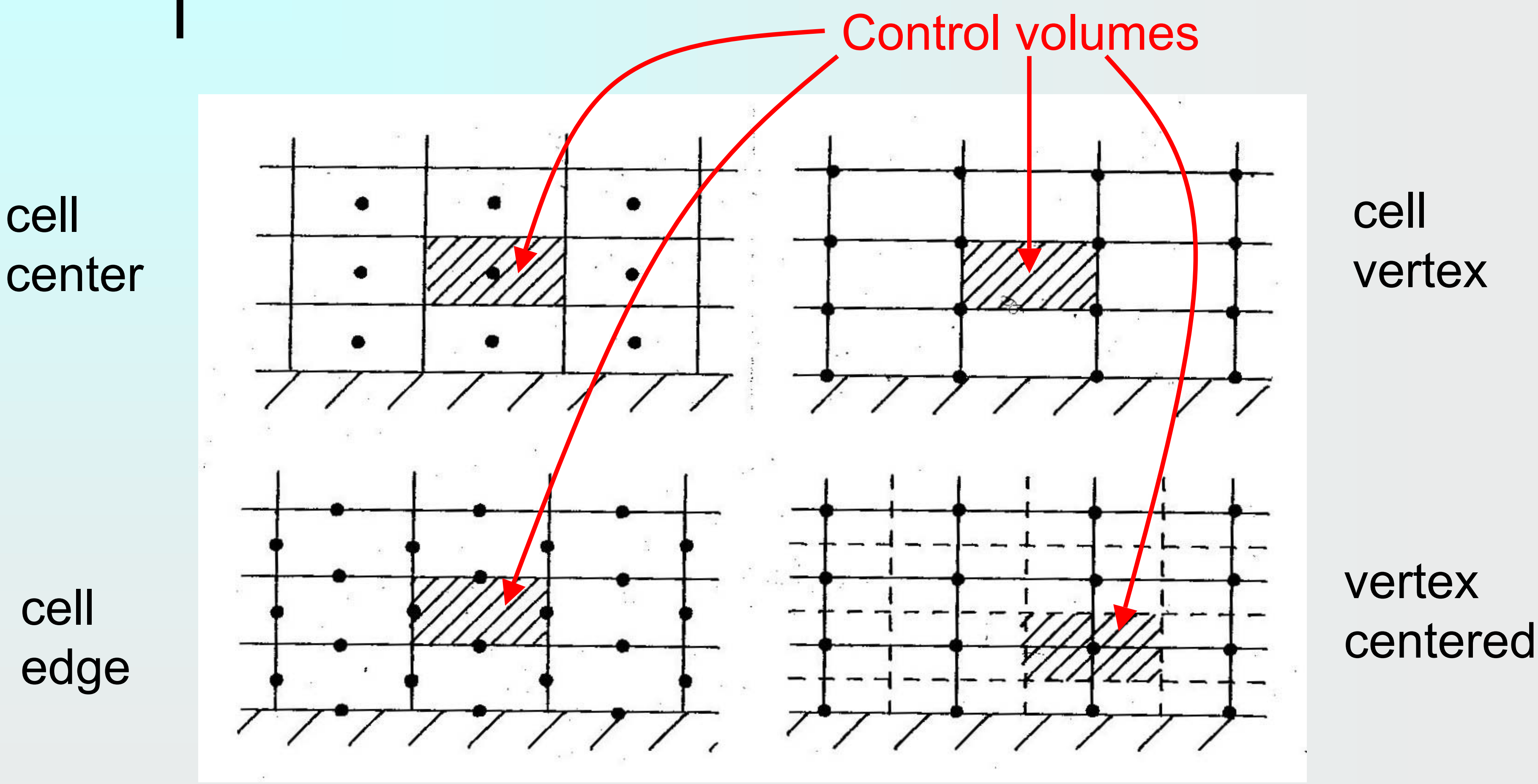
- Because of the walls:
  $\mathbf{v} \cdot \mathbf{n} = 0 \qquad \nabla\Phi \cdot n = 0$

$$\int_V \frac{\partial \Phi}{\partial t} dV + \int_S \rho \Phi \, \mathbf{v} \cdot \mathbf{n} \, dS = \int_S \Gamma \nabla \Phi \cdot \mathbf{n} \, dS \quad \text{for any } V \subseteq \Omega$$

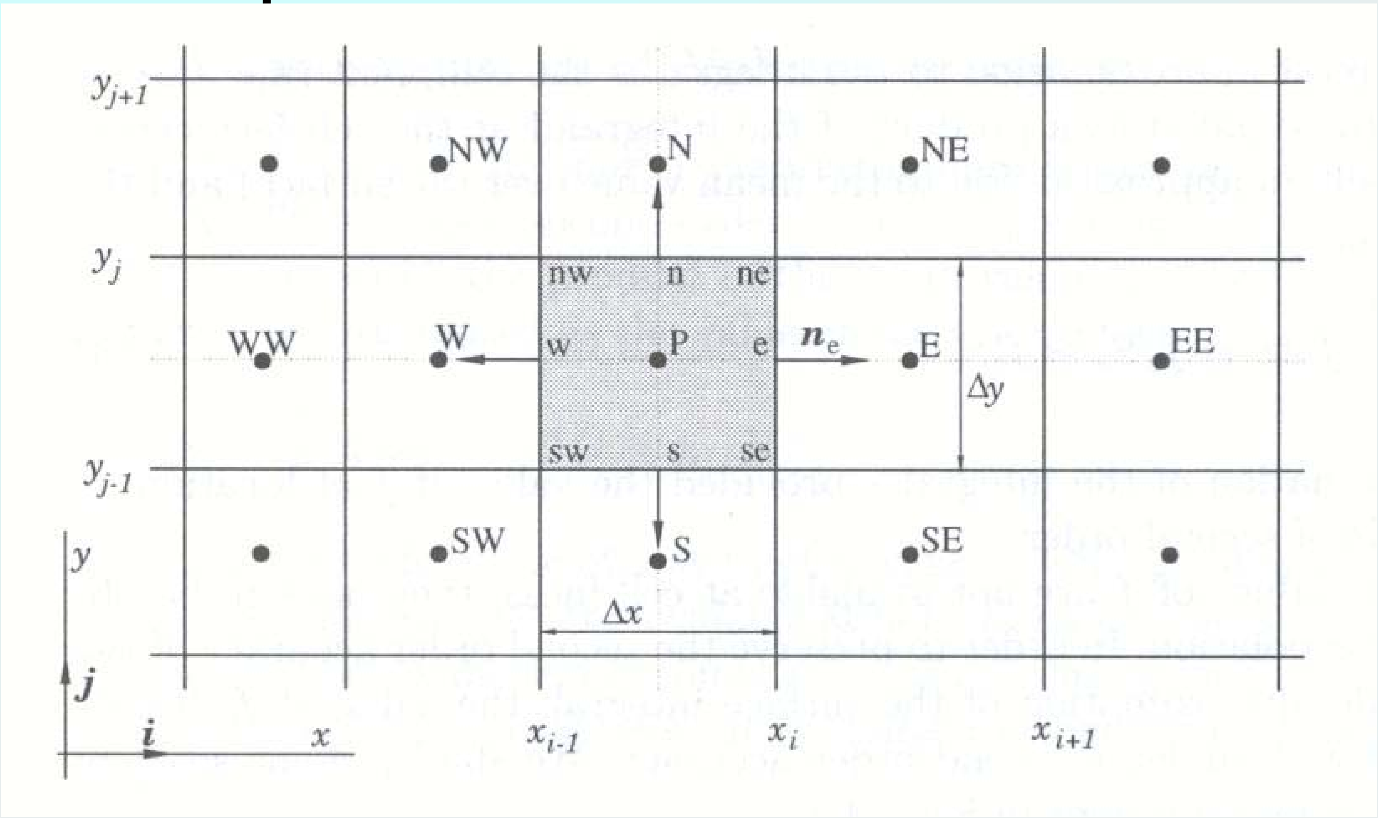
- Analytical solution: $\int_\Omega \Phi \, dV = \text{const}$

- Numerical solution: a discrete equivalent of

Different finite volume schemes



Numerical discretization

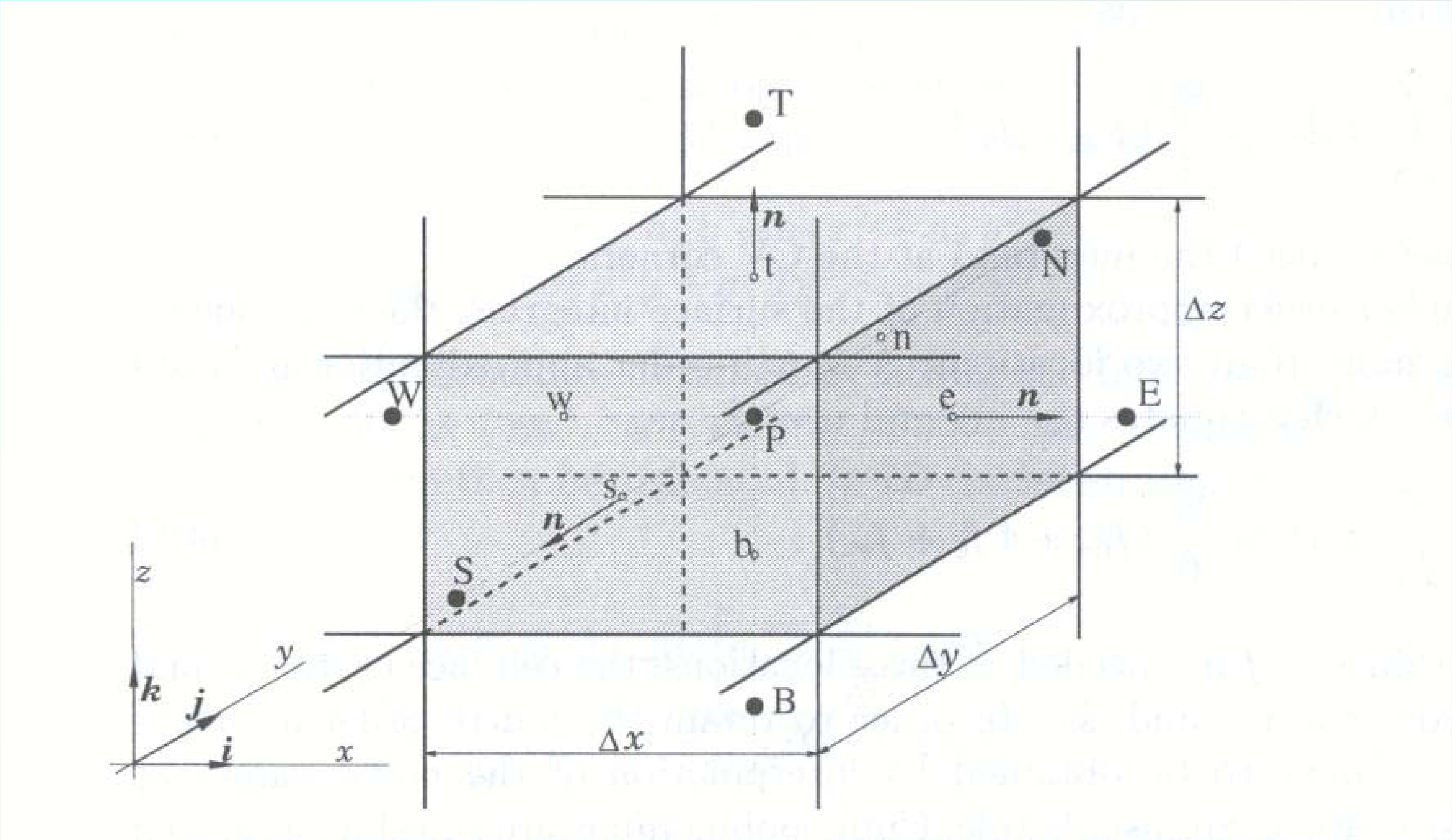


(Cell centered approach)

1. Quadrature rule
2. Interpolation scheme

$$\int_S \rho\Phi\, \mathbf{v}\cdot\mathbf{n}\, dS = \int_S \Gamma\nabla\Phi\cdot\mathbf{n}\, dS + \int_V s\, dV \quad \text{for any } V \subseteq \Omega$$

Three-dimensional case



Fluxes

- Flux $\mathbf{f}(\Phi)$

convective flux $\mathbf{f}^c$

diffusive flux $\mathbf{f}^d$

$$\mathbf{f} = \rho\Phi\,\mathbf{v} - \Gamma\nabla\Phi$$

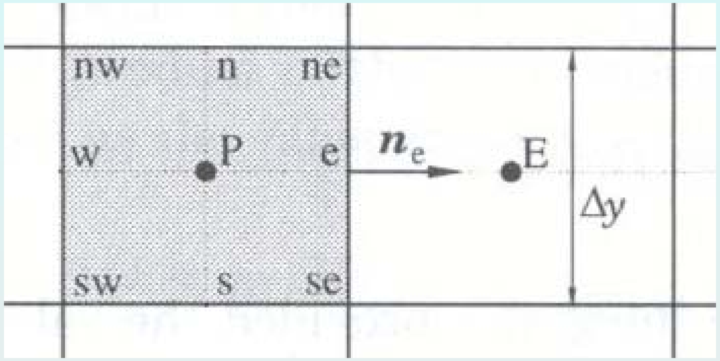
- Integrated flux $F(\Phi)$

$$F = \int_S \mathbf{f}\cdot\mathbf{n}\,dS = \int_S \rho\Phi\,\mathbf{v}\cdot\mathbf{n}\,dS - \int_S \Gamma\nabla\Phi\cdot\mathbf{n}\,dS$$

- Equation

$$\int_S \mathbf{f}\cdot\mathbf{n}\,dS = \int_V s\,dv$$

Approximation of surface integrals



- Midpoint rule (2nd order)

$$F_e = \int_{S_e} \mathbf{f} \cdot \mathbf{n} \, dS \approx f_e S_e$$

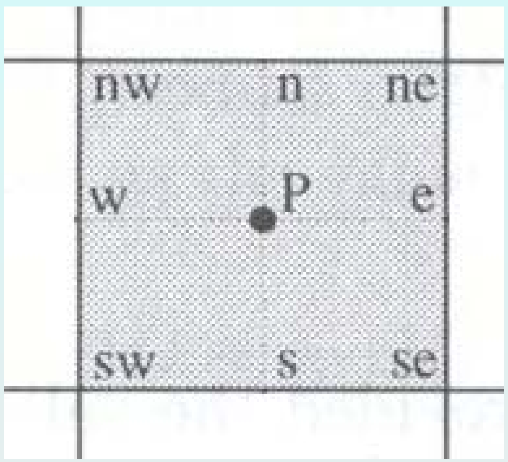
- Trapezoidal rule (2nd order)

$$F_e = \int_{S_e} \mathbf{f} \cdot \mathbf{n} \, dS \approx S_e \frac{1}{2}(f_{ne} + f_{se})$$

- Simpson's rule (4th order)

$$F_e = \int_{S_e} \mathbf{f} \cdot \mathbf{n} \, dS \approx S_e \frac{1}{6}(f_{ne} + 4f_e + f_{se})$$

Approximation of volume integrals



- 2nd order formula

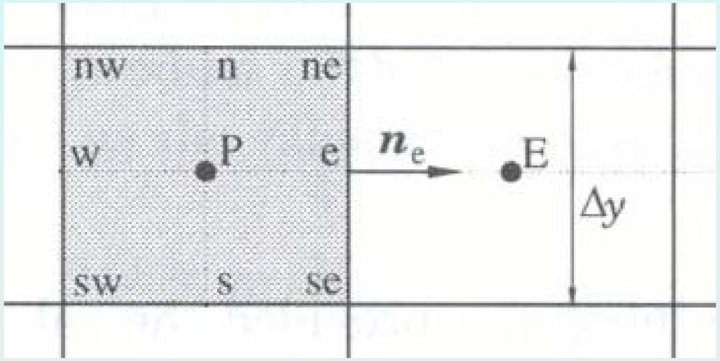
$$\int_V s\,dV \approx s_P\,\Delta x\,\Delta y$$

- 4th order formula (uniform Cartesian grid)

$$\int_V s\,dV \approx \frac{\Delta x\,\Delta y}{36}(16 s_P + 4 s_s + 4 s_n + 4 s_e + 4 s_w + s_{se} + s_{sw} + s_{ne} + s_{nw})$$

Interpolation schemes: UDS

1st order accuracy



- Upwind differencing scheme (UDS)

$$\Phi_e = \begin{cases} \Phi_P & \text{if } (\mathbf{v}\cdot\mathbf{n})_e > 0 \\ \Phi_E & \text{if } (\mathbf{v}\cdot\mathbf{n})_e < 0 \end{cases}$$

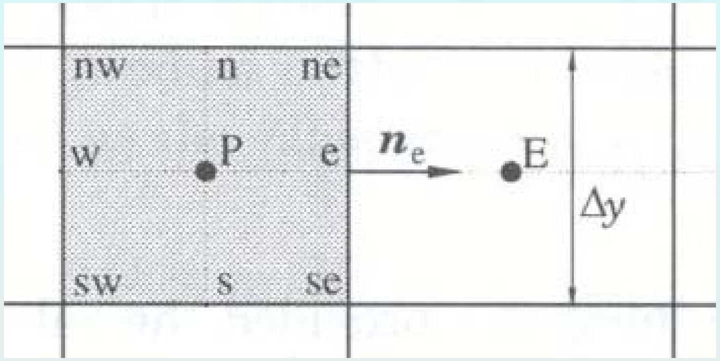
☺ Never oscillatory solutions

☹ Artificial diffusion:

$$f_e^d = \Gamma_e^{num}\left(\frac{\partial \Phi}{\partial x}\right)_e = \frac{\rho_e v_e \Delta x}{2}\left(\frac{\partial \Phi}{\partial x}\right)_e$$

Interpolation schemes: CDS

2nd order accuracy



- Centered diff. scheme (CDS)

$$\Phi_e \approx \Phi_E \frac{x_e - x_P}{x_E - x_P} + \Phi_P \frac{x_E - x_e}{x_E - x_P}$$ f^c

$$\left(\frac{\partial \Phi}{\partial x}\right)_e \approx \frac{\Phi_E - \Phi_P}{x_E - x_P}$$ f^d

☺ More accurate than UDS

☹ May produce oscillatory solutions

High order interpolation schemes

- QUICK (Quadratic Upwind Interpolation for Convective Kinematics)

$$\Phi_e \approx \frac{6}{8}\Phi_P + \frac{3}{8}\Phi_E - \frac{1}{8}\Phi_W - \frac{3}{48}(\Delta x)^3 \left(\frac{\partial^3 \Phi}{\partial x^3}\right)_P$$

- 4th order CDS

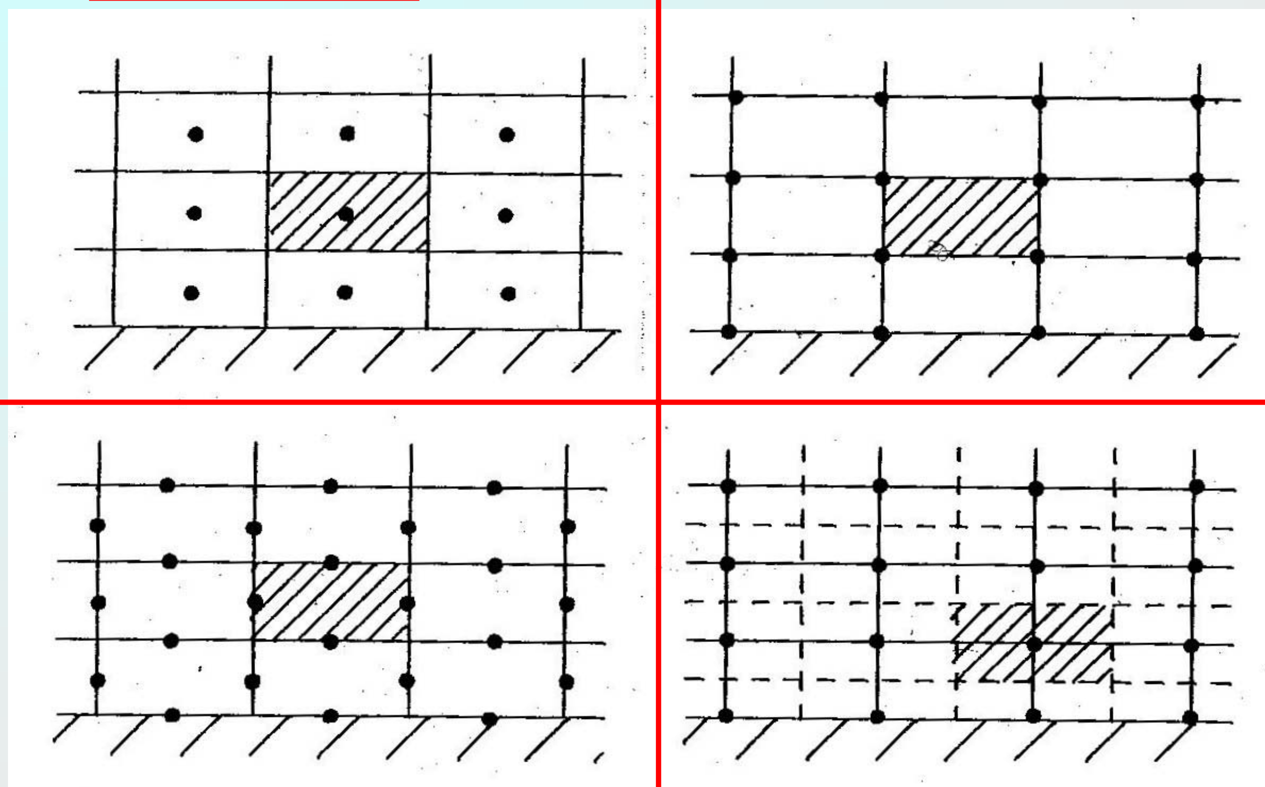
$$\Phi_e \approx \frac{1}{48}\left(27\Phi_P + 27\Phi_E - 3\Phi_W - 3\Phi_{EE}\right)$$

$$\left(\frac{\partial \Phi}{\partial x}\right)_e \approx \frac{1}{24\Delta x}\left(27\Phi_E - 27\Phi_P + \Phi_W - \Phi_{EE}\right)$$

Remark on different schemes

☺ diffusive | ☺ convective

cell center | cell vertex

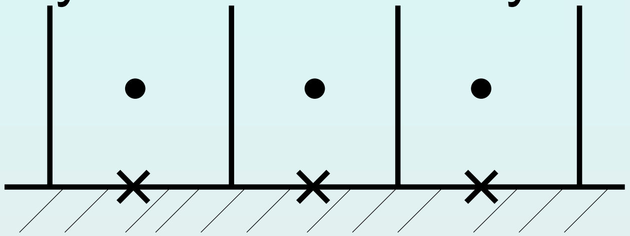


cell edge | vertex centered

☺ convective | ☺ diffusive

Linear system and boundary conditions

- Equations for Φ_P form a linear system
- System is closed by boundary conditions



e.g. One side differences for diffusive fluxes

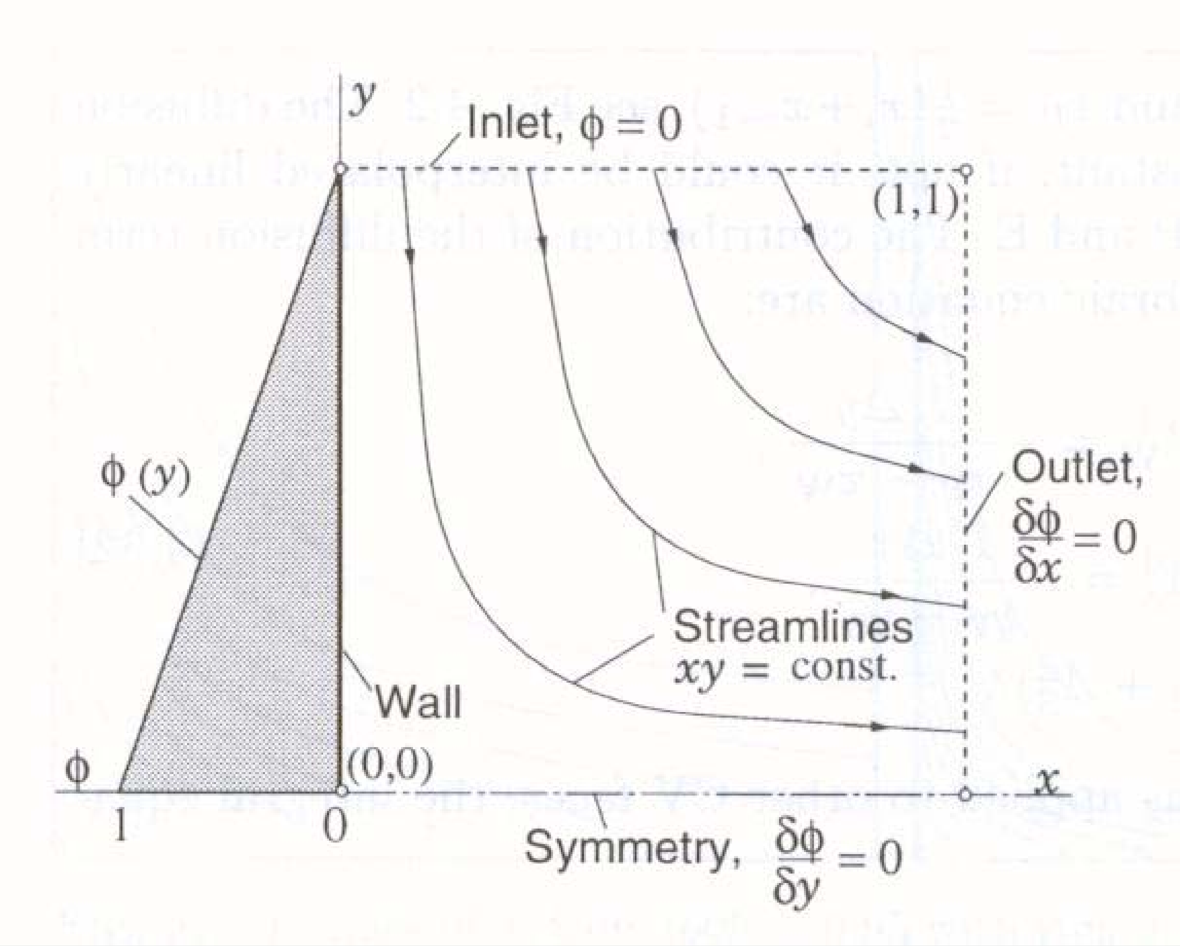
- Example:
  - 2D, cell centered, midpoint rule + 2nd CDS | 1st UDS: pentadiagonal system

Deferred correction

- High order schemes → Large computational molecule
  - 2D, Simpson rule + 4th order CDS:
    each flux depends on 15 nodal values
- Large computational molecule → Expensive solution of linear system
- Idea: combine low and high order approximations
  - High order approximation are only computed explicitly

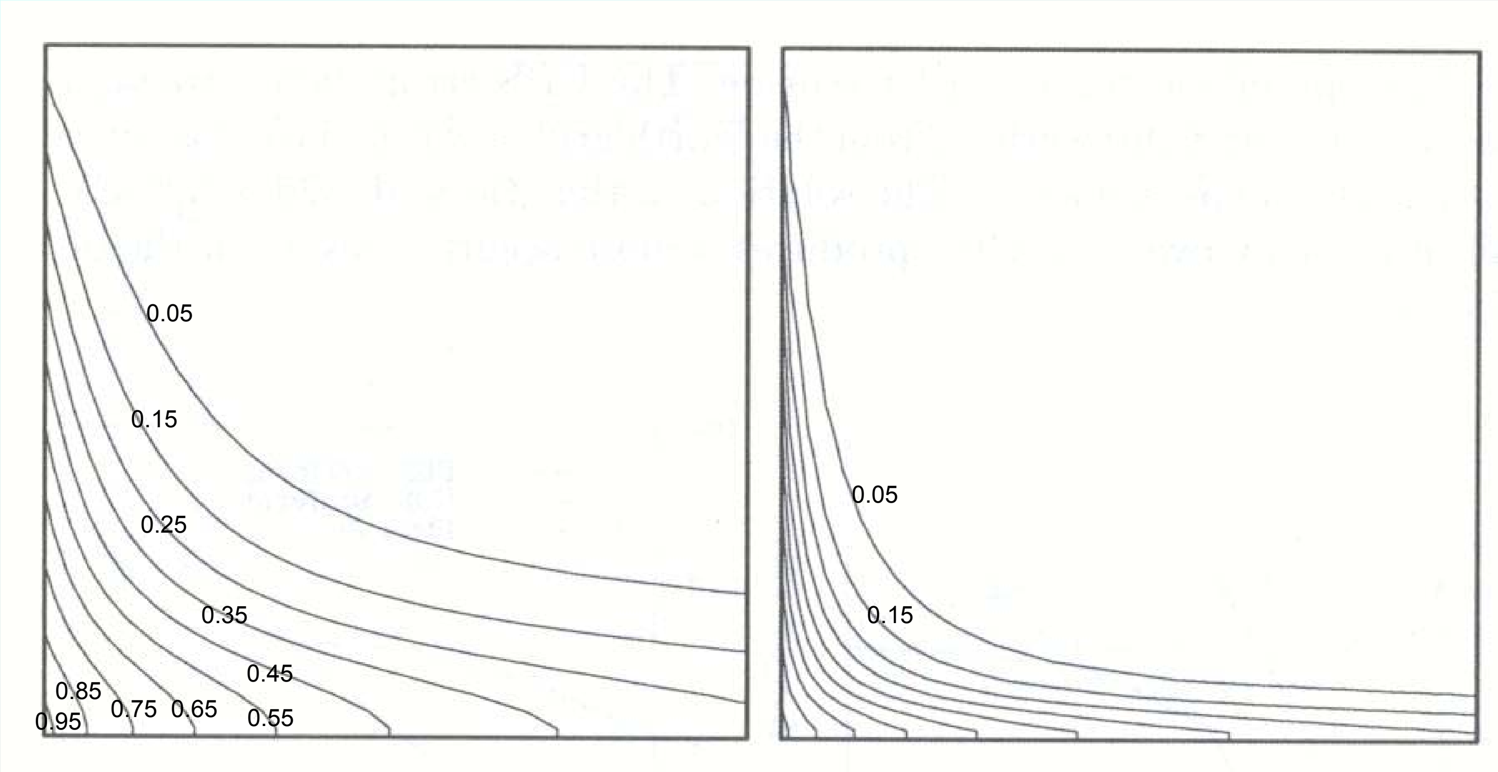
$$F_e = F_e^{LOW} + \left(F_e^{HIGH} - F_e^{LOW}\right)_{old}$$

Example



- Velocity field: $\mathbf{v}=(v_x,v_y) = (x,-y)$
- Density $\rho = 1$
- Test UDS and CDS with midpoint rule and cell centered

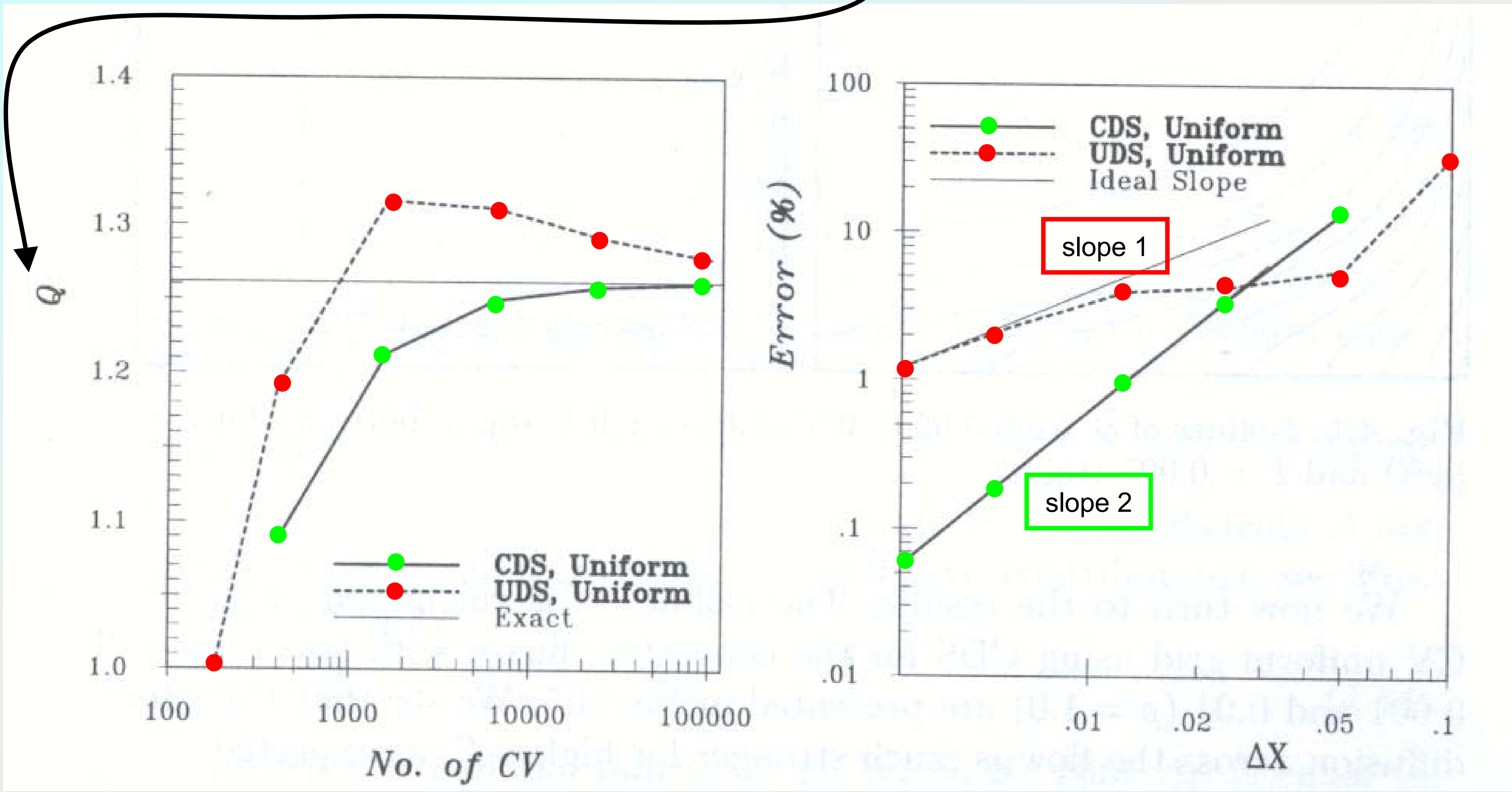
Isolines of Φ for different Γ



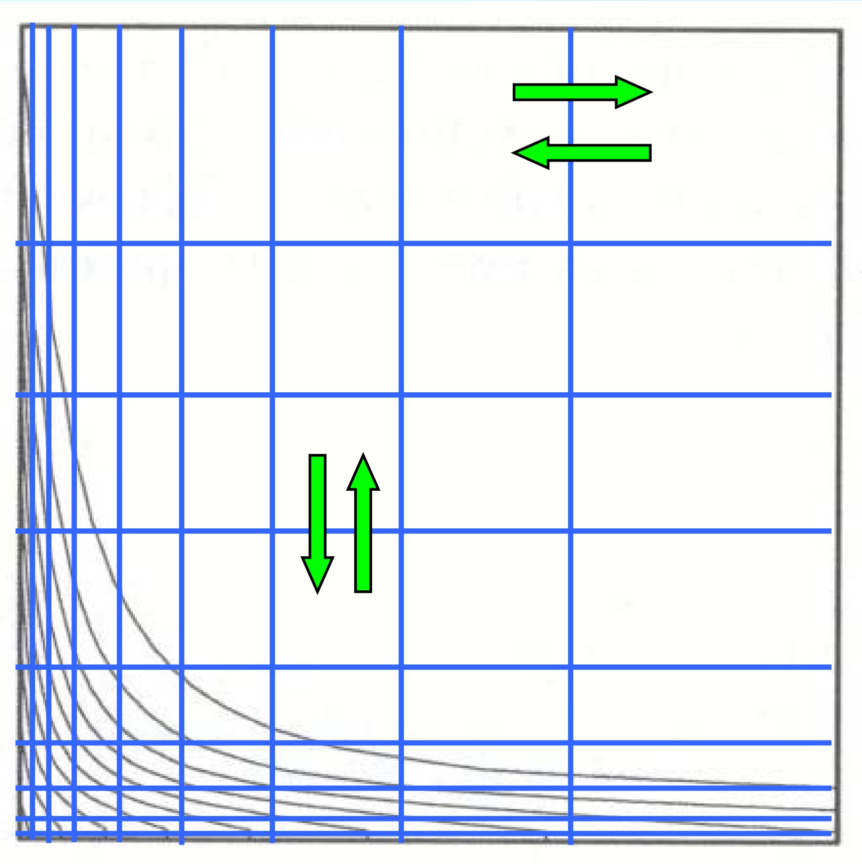
$\Gamma = 10^{-2}$ $\Gamma = 10^{-3}$

Convergence

Q = Integrated flux through the west side of the domain



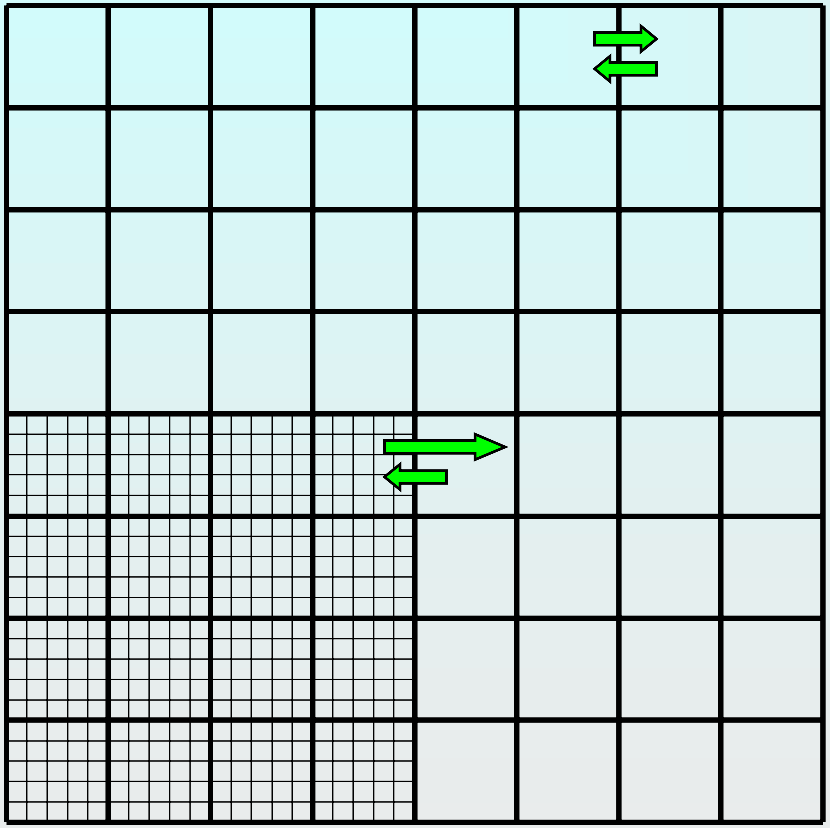
Different grid types



- Tensor product grid
- By construction, in neighboring control volumes influxes and outfluxes are balanced
  - sum of discrete conservation laws

$\Gamma = 10^{-3}$

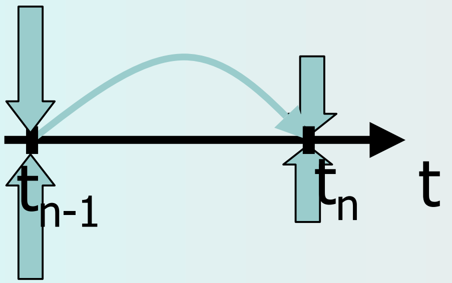
Composite grid

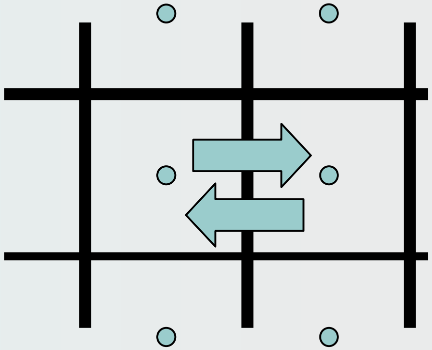


- Balance of fluxes across the interface coarse-fine grid is not guaranteed
- In Local Defect Correction (LDC)
  - iterative improvement between coarse and fine grid solution
  - in the limit: balance of fluxes everywhere

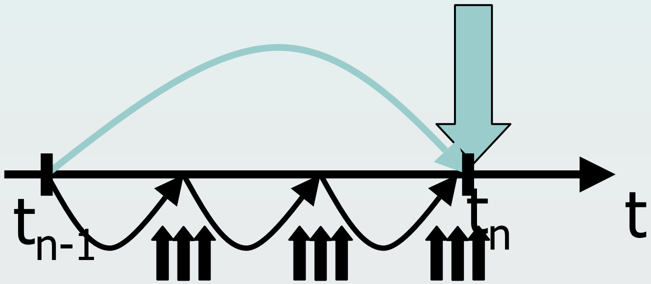
Time-dependent problem

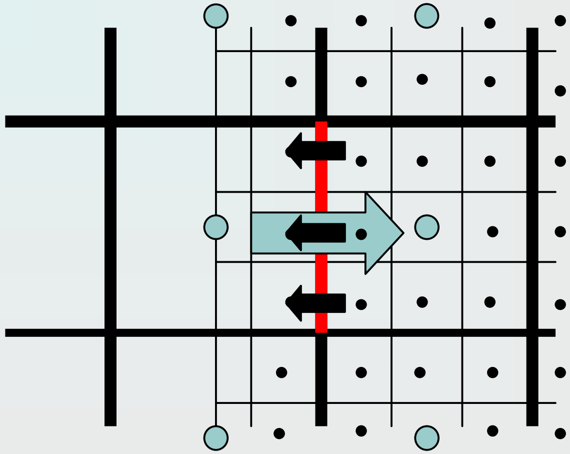
- Standard grids
  - sum of conservation laws also in time





- Composite grid with different rates for time integration
  - LDC: balance preserved





Conclusions

- The main ideas behind the finite volume methods were introduced
- Schemes for quadrature and interpolation were discussed
- Some issues about conservation on different grid types were addressed