Strain and deformation

a global overview

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Seminar on Continuum Mechanics

Continuum mechanics

Continuum mechanics

Definition

Continuum mechanics is a branch of mechanics concerned with the stresses in solids, liquids and gases and the deformation or flow of these materials.

A continuum disregards the molecular structure of matter and pictures it as being without gaps or empty spaces.

Continuum mechanics

Continuum mechanics		

Seminar topics

- Stress
- Strain and deformation
- General principles

Continuum mechanics

Continuum mechanics	Solid mechanics	
	Fluid mechanics	

Definition

- Solid mechanics deals with solid materials. A solid has a defined rest shape and can support shear stresses.
- Fluid mechanics deals with fluids (both liquids and gases). A fluid takes the shape of its container and cannot support shear stresses.

Continuum mechanics

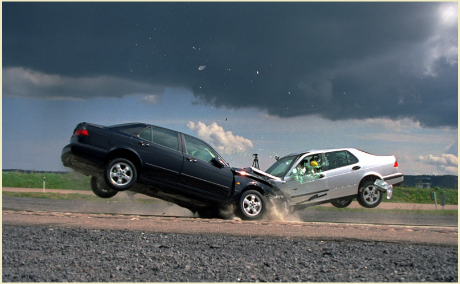
Continuum mechanics	Solid mechanics	Elasticity
		Plasticity
	Fluid mechanics	

Definition

- Elasticity describes materials that return to their rest shape after an applied stress.
- Plasticity describes materials that permanently deform (change their rest shape) after a large enough applied stress.

Continuum mechanics

Continuum mechanics	Solid mechanics	Elasticity
		Plasticity
	Fluid mechanics	



Continuum mechanics

Continuum mechanics	Solid mechanics	Elasticity
		Plasticity
	Fluid mechanics	non-Newtonian fluids
		Newtonian fluids

Definition

- non-Newtonian fluids are fluids in which the viscosity changes with the applied shear stress.
- Newtonian fluids are fluids in which the viscosity is constant.

Continuum mechanics

Continuum mechanics	Solid mechanics	Elasticity
		Plasticity
	Fluid mechanics	non-Newtonian fluids
		Newtonian fluids

Continuum mechanics

Continuum mechanics	Solid mechanics	Elasticity
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	Fluid mechanics	non-Newtonian fluids
		Newtonian fluids

Seminar topics

- Constitutive equations
- Linearized theory of elasticity
- Fluid mechanics
- . . .

Outline

1. Kinematics of a continuous medium
   - Continuum configuration
   - Motion and material derivatives
   - Deformation and strain
   - Rate of deformation and vorticity
   - Polar decomposition
2. Linear deformation and strain theory
   - Linear deformation and strain
   - Principal strains and invariants
   - Compatibility conditions

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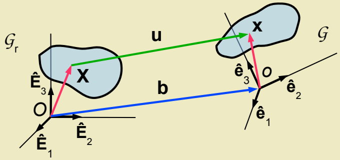
Continuum configuration

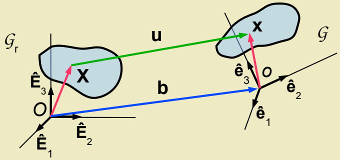


$$\mathbf{X} = X_1\hat{\mathbf{E}}_1 + X_2\hat{\mathbf{E}}_2 + X_3\hat{\mathbf{E}}_3 \qquad \mathbf{x} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3$$

Definition

- Let $\mathcal{B}$ be a 3-dimensional, continuous, material body and let $P \in \mathcal{B}$ be a material point.
- Let $\mathcal{G} \subset \mathbb{R}^3$ be a configuration of $\mathcal{B}$ at time t and $\mathcal{G}_r \subset \mathbb{R}^3$ a reference configuration.

Continuum configuration



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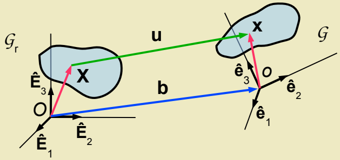
Definition

- Let $\mathbf{X} \in \mathcal{G}_r$ be the position of material point P in the reference configuration with respect to origin O.
- Let $\mathbf{x} \in \mathcal{G}$ be the position of material point P at time t with respect to origin o.

Continuum configuration



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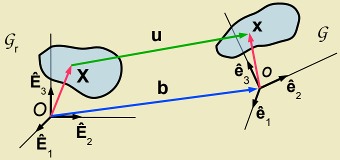
Definition

Then two bijective mappings exist

- $\mathbf{\Phi} : \{(\mathbf{X}, t) \,|\, \mathbf{X} \in \mathcal{G}_r, t \in \mathbb{R}\} \to \{\mathbf{x} | \mathbf{x} \in \mathcal{G}\} : \mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t),$
- $\mathbf{\Psi} : \{(\mathbf{x}, t) \,|\, \mathbf{x} \in \mathcal{G}, t \in \mathbb{R}\} \to \{\mathbf{X} | \mathbf{X} \in \mathcal{G}_r\} : \mathbf{X} = \mathbf{\Psi}(\mathbf{x}, t).$

Continuum configuration



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Definition

The displacement vector **u** links the material coordinates **X** with the spatial coordinates **x** through

$$\mathbf{u} = \mathbf{b} + \mathbf{x} - \mathbf{X}.$$

Often in continuum mechanics it is possible to consider both coordinate systems superimposed and then $\mathbf{b} = \mathbf{0}$.

Continuum configuration

Example

- Rigid body motion

$$\rightarrow \mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{X},$$

$$\rightarrow \mathbf{X} = \mathbf{\Psi}(\mathbf{x}, t) = \mathbf{Q}^T(t)(\mathbf{x} - \mathbf{c}(t)).$$

- Uniform dilatation

$$\rightarrow \mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t) = (1 + \epsilon(t))\mathbf{X},$$

$$\rightarrow \mathbf{X} = \mathbf{\Psi}(\mathbf{x}, t) = \frac{1}{1+\epsilon(t)}\mathbf{x}.$$

- Note that this formulation excludes crack formation

Description of motion

Definition

1. Material description, whose independent variables are the particle P and the time t.
2. Referential description, whose independent variables are the position $\mathbf{X}$ of the particle in a reference configuration and the time t (Lagrangian description).
3. Spatial description, whose independent variables are the present position $\mathbf{x}$ occupied by the particle at time t and the present time t (Eulerian description).
4. Relative description, whose independent variables are the present position $\mathbf{x}$ occupied by the particle and a variable time τ.

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Material and local time derivatives

Definition

Consider an arbitrary field quantity **F**. The material time derivative (denoted with $\frac{d}{dt}$) and the local time derivative (denoted with $\frac{\partial}{\partial t}$) are given by

$$\frac{d\mathbf{F}}{dt} := \frac{\partial \tilde{\mathbf{F}}(\mathbf{X}, t)}{\partial t}, \qquad \frac{\partial \mathbf{F}}{\partial t} := \frac{\partial \bar{\mathbf{F}}(\mathbf{x}, t)}{\partial t}.$$

After applying the chain rule the following relation is found

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{F},$$

where $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ is the instantaneous velocity of the particle (material derivative of the particle's position).

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Material and local time derivatives

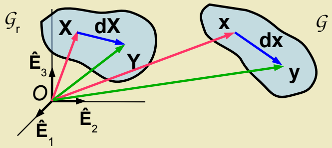
Example

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

Applying the material derivative operator on

- Density ρ : $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho.$
- Displacement $\mathbf{u}$: $\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{u}.$
- Velocity $\mathbf{v}$: $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}.$

Deformation and displacement gradients

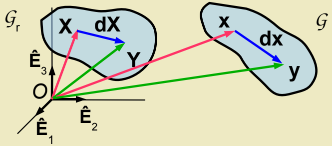


Definition

A motion where the shape and/or volume of $\mathcal{B}$ is changed is called a deformation. In a deformation the distance between two material points changes

$$
\begin{aligned}
\mathbf{x} &= \boldsymbol{\Phi}(\mathbf{X}, t), \\
\mathbf{y} &= \boldsymbol{\Phi}(\mathbf{Y}, t) = \boldsymbol{\Phi}(\mathbf{X}, t) + \frac{\partial \boldsymbol{\Phi}(\mathbf{X}, t)}{\partial \mathbf{X}} \mathbf{dX} + O(|\mathbf{dX}|) \\
&=: \mathbf{x} + \mathcal{F}(\mathbf{X}, t)\mathbf{dX} + O(|\mathbf{dX}|).
\end{aligned}
$$

Deformation and displacement gradients



Definition

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$$\mathbf{dx} \approx \mathcal{F}\mathbf{dX},$$

where $\mathcal{F} = \frac{\partial \boldsymbol{\Phi}(\mathbf{X},t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ is the material deformation gradient.

Also $\mathcal{G} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathcal{F} - \mathcal{I}$ is the material displacement gradient.

Deformation and strain tensors

Definition

Because $\mathcal{F}$ still includes rigid body rotation it is not a direct measure for deformation. Therefore look at the change of length of a line-element between two material points

$$\begin{aligned} |\mathbf{dx}|^2 &= (\mathbf{dx}, \mathbf{dx}) = (\mathcal{F}\mathbf{dX}, \mathcal{F}\mathbf{dX}) \\ &= (\mathcal{F}^T\mathcal{F}\mathbf{dX}, \mathbf{dX}) := (\mathcal{C}\mathbf{dX}, \mathbf{dX}), \end{aligned}$$

where $\mathcal{C} = \mathcal{F}^T\mathcal{F}$ is the right Cauchy-Green deformation tensor.

A deformation quantity which becomes zero when there is no deformation present is the Lagrangian strain tensor

$$\mathcal{E} = \frac{1}{2}(\mathcal{C} - \mathcal{I}) = \frac{1}{2}(\mathcal{G} + \mathcal{G}^T + \mathcal{G}^T\mathcal{G}).$$

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Rate of deformation and spin tensor

Definition

In solid mechanics the deformation and displacement gradients play an important role. In fluid mechanics it is often the gradient of the velocity that is important

$$\mathcal{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{d\mathcal{F}}{dt}\mathcal{F}^{-1} =: \mathcal{D} + \mathcal{W},$$

where $\mathcal{D} = \frac{1}{2}(\mathcal{L} + \mathcal{L}^T)$ is the rate of deformation tensor
and $\mathcal{W} = \frac{1}{2}(\mathcal{L} - \mathcal{L}^T)$ is the spin tensor.

Moreover, the vorticity vector $\mathbf{w} = \frac{1}{2}\nabla \times \mathbf{v}$ is associated with the anti-symmetric tensor $\mathcal{W}$.

Stretch and rotation

Definition

A polar decomposition of an arbitrary, nonsingular second-order tensor is given by the product of a *symmetric positive-definite* tensor and an *orthogonal* tensor. For the deformation gradient this means

$$\mathcal{F} = \mathcal{R}\mathcal{U} = \mathcal{V}\mathcal{R},$$

where

- $\mathcal{R}$ is the rotation tensor
- $\mathcal{U}$ is the right stretch tensor
- $\mathcal{V}$ is the left stretch tensor

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   - Compatibility conditions

Linear deformation

Definition

In linear deformation theory the displacement gradients are small compared to unity

$$||\mathcal{G}|| = \left|\left|\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right|\right| =: \varepsilon \ll 1.$$

In linear deformation theory all $O(\varepsilon^2)$ terms are neglected. A consequence of this is that the material and spatial displacement gradients are very nearly equal

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\left(\mathcal{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right) = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}(1 + O(\varepsilon)).$$

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$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}.$$

Linear strain tensor

Definition

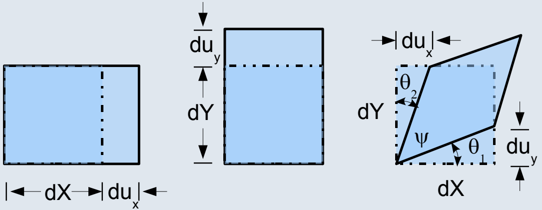
Neglecting the higher order terms in the Lagrangian strain tensor gives the linear Lagrangian strain tensor

$$
\begin{aligned}
\mathcal{E} &= \frac{1}{2}(\mathcal{G} + \mathcal{G}^T + \mathcal{G}^T\mathcal{G}) \\
&= \frac{1}{2}(\mathcal{G} + \mathcal{G}^T + O(\varepsilon^2)) \\
\mathcal{E}_l &:= \frac{1}{2}(\mathcal{G} + \mathcal{G}^T) = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^T\right).
\end{aligned}
$$

2D interpretation of linear strain tensor

Example

- Uniaxial extension in x-direction: $\varepsilon_{xx} \approx \frac{du_x}{dX}$.
- Uniaxial extension in y-direction: $\varepsilon_{yy} \approx \frac{du_y}{dY}$.
- Pure shear without rotation: $\gamma_{xy} = \frac{\pi}{2} - \psi = \theta_1 + \theta_2$,
$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} \approx \frac{1}{2}\left(\frac{du_x}{dY} + \frac{du_y}{dX}\right).$$



Principal strains and invariants

Properties

Several properties hold for the symmetric, second-order linear strain tensor

- The principal strain direction is a direction for which the orientation of an element at a given point is not altered by a pure strain deformation (no shear strain component).
- The principal strain values $(\epsilon_1, \epsilon_2, \epsilon_3)$ are the unit relative displacements (normal strain components) that occur in the principal directions.

Principal strains and invariants

Properties

Several properties hold for the symmetric, second-order linear strain tensor

- The invariants are given by

$$
\begin{aligned}
I_{\mathcal{E}_l} &= \operatorname{tr} \mathcal{E}_l = \epsilon_1 + \epsilon_2 + \epsilon_3, \\
II_{\mathcal{E}_l} &= \epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1, \\
III_{\mathcal{E}_l} &= \det \mathcal{E}_l = \epsilon_1\epsilon_2\epsilon_3.
\end{aligned}
$$

Principal strains and invariants

Properties

Several properties hold for the symmetric, second-order linear strain tensor

- An additive decomposition consisting of a spherical tensor and deviator tensor

$$\mathcal{E}_l = \epsilon_M \mathcal{I} + (\mathcal{E}_l - \epsilon_M \mathcal{I}),$$

where $\epsilon_M = (\epsilon_1 + \epsilon_2 + \epsilon_3)/3$ is the mean normal strain.

The deviator tensor is associated with shear deformation for which the cubical dilatation vanishes.

Compatibility conditions

Definition

If the strain components are given, the symmetric linear strain matrix may be viewed as a system of *six* PDEs for determining the *three* components of the displacement vector **u**.

For a solution to exist, a necessary and sufficient condition is given by the compatibility relations

$$
\begin{aligned}
2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} &= \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2}, \\
\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x}\Big(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\Big).
\end{aligned}
$$

Summary

Strain and deformation: *a global overview*

The kinematics of a general continuous medium have been discussed. Several important quantities have been introduced

- Material and spatial coordinates
- Deformation and strain
- Rate of deformation and vorticity

Linear deformation theory simplifies the general theory on the assumption that the displacement gradients are small.

Summary

Strain and deformation: *a global overview*

The kinematics of a general continuous medium have been discussed. Several important quantities have been introduced

- Material and spatial coordinates
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- Rate of deformation and vorticity

Linear deformation theory simplifies the general theory on the assumption that the displacement gradients are small.

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