Fluid Mechanics

a complete theory

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5th Seminar on Continuum Mechanics

3th May 2006

Continuum Mechanics

Seminar Outline

- Stress
- Strain and deformation
- General principles
- Constitutive equations
- Fluid Mechanics
- ...

Fluid Mechanics

Continuum Mechanics	Solid Mechanics	
	Fluid Mechanics	Non-Newtonian Fluid
		Newtonian Fluid

Definitions:

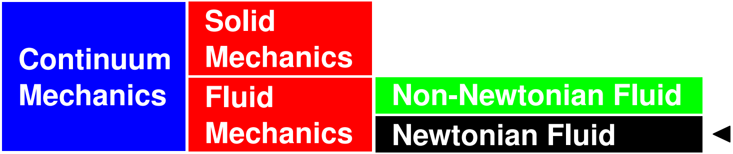
- Fluid mechanics deals with fluids (both liquids and gases). A fluid takes the shape of its container and cannot support shear stresses.
- Newtonian fluids are fluids in which the viscosity is constant.

Fluid Mechanics

Continuum Mechanics

- **Solid Mechanics**
- **Fluid Mechanics**
  - **Non-Newtonian Fluid**
  - **Newtonian Fluid** ◀

Fluid Mechanics



The general problem

- Steady-flow boundary value problems:
  - $\Rightarrow$ time-independent flow problems.
- Transient problems:
  - $\Rightarrow$ starting-up problem in a pipe;
  - $\Rightarrow$ propagation of sound waves through air or water;
  - $\Rightarrow$ generally more difficult to solve.

Outline

The field equations of a Newtonian fluid

General theorems (5)	Constitutive equations (11)
$\frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$	$F(p, \rho, \theta) = 0$
$\vec{\nabla} \cdot T + \rho \vec{b} = \rho \frac{d\vec{v}}{dt}$	$T = (\mathrm{tr}(D)\lambda - p)\, I + 2\mu D$
$T : D + \phi = \vec{\nabla} \cdot \vec{q} + \rho \frac{du}{dt}$	$\vec{q} = -k \vec{\nabla} \theta$
	$u = u(\theta, \rho)$

16 unknown variables	
$\vec{v}$: velocity	ρ: density
p: pressure	u: internal energy
θ: temperature	T: stress tensor
	$\vec{q}$: heat flux

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$\frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$	$F(p, \rho, \theta) = 0$
$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} p + (\lambda + \mu)\vec{\nabla}(\vec{\nabla} \cdot \vec{v}) + \mu \Delta \vec{v} + \rho \vec{b}$	
$\rho \frac{du}{dt} = \vec{\nabla} \cdot (k \vec{\nabla} \theta) - p \vec{\nabla} \cdot \vec{v} + \phi + 2W_D$	
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7 unknown variables

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Remarks

- The equations are only valid in laminar-flow situations and not for turbulent flow.

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Remarks

- Non-linearities appear in the inertial acceleration terms:

$$\rho \frac{d\vec{v}}{dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v}.$$

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Remarks

- For barotropic flows: $F(p, \rho) = 0$.
  - $\Rightarrow$ The five unknowns $\vec{v}$, p and ρ can be determined using only Navier-Stokes, the continuity equation and the barotropic equation of state.

Simplifications

Navier-Stokes equations of motion

- Compressible fluid with Stokes condition: $\lambda = -\frac{2}{3}\mu$,

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\vec{\nabla} p + \mu \Delta \vec{v} + \rho \vec{b} + \frac{\mu}{3} \vec{\nabla}(\vec{\nabla} \cdot \vec{v}).$$

- Incompressible fluid: $\vec{\nabla} \cdot \vec{v} = 0$,

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\vec{\nabla} p + \mu \Delta \vec{v} + \rho \vec{b},$$

- For irrotational flows: $\vec{\nabla} \times \vec{v} = 0$,

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\vec{\nabla} p + \rho \vec{b} \quad \text{(perfect fluid)}.$$

Dimensional analysis

Buckingham π theorem

If we have m physical parameters $q_1, \dots, q_m$, expressed in k independent physical units, satisfying: $F(q_1, \dots, q_m) = 0$, then there are $n = m - k$ independent dimensionless parameters $\pi_1, \dots, \pi_n$ satisfying: $G(\pi_1, \dots, \pi_n) = 0$.

$\Rightarrow$ Here we have 7 independent dimensionless parameters.

Dimensional analysis

- Rescale the variables with their characteristic values:

$$x = Lx', \quad t = Tt'$$

$$\vec{v} = V\vec{v}', \quad p = p_0 p', \quad \rho = \rho_0 \rho', \quad \theta = \theta_0 \theta'.$$

Dimensional analysis

Dimensionless numbers

$E = p_0/(\rho_0 V^2)$	Euler	$M = V/c$	Mach
$Re = \rho_0 VL/\mu$	Reynolds	$Pe = \rho_0 c_p VL/k$	Péclet
$Fr = V^2/(gL)$	Froude	$\gamma = c_p/c_v$	
$Sr = L/(TV)$	Strouhal		

Navier-Stokes equation

- Suppose $\overrightarrow{b} = -\overrightarrow{g}$.
- The rescaled equations of motion read:

$$Sr\rho' \frac{\partial \overrightarrow{v}'}{\partial t'} + \rho' \overrightarrow{v}' \cdot \nabla' \overrightarrow{v}' = -E\overrightarrow{\nabla}' p' + \frac{1}{Re}\left(\Delta' \overrightarrow{v}' + \frac{1}{3}\overrightarrow{\nabla}'(\overrightarrow{\nabla}' \cdot \overrightarrow{v}')\right) + \frac{1}{Fr}\overrightarrow{g}'.$$

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Continuity equation

- The rescaled continuity equation reads:

$$Sr\frac{\partial \rho'}{\partial t'} + \overrightarrow{\nabla}' \cdot (\rho' \overrightarrow{v}') = 0.$$

Dimensional analysis

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Energy equation

- For a perfect gas $\mathrm{d}u = c_v \mathrm{d}\theta$.
- The rescaled energy equation becomes:

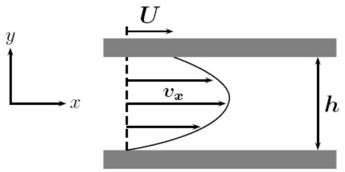
$$Sr\rho' \frac{\partial \theta'}{\partial t'} + \rho' \vec{v}' \cdot \vec{\nabla}' \theta' = \frac{\gamma}{Pe} \Delta' \theta' - (\gamma - 1) p' \vec{\nabla}' \cdot \vec{v}' + \frac{2(\gamma - 1)}{ERe} W_D'.$$

Special cases

Steady (incompressible) laminar flow between parallel plates

- Non-linearities disappear due to incompressibility and geometry:

$$\left.\begin{array}{r} -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2} = 0, \\ -\frac{\partial p}{\partial y} - \rho g = 0, \end{array}\right\} \rightarrow \left\{ \begin{array}{l} p = -\rho g z + C x + \frac{\mu}{h} U - \frac{h}{2} C, \\ \mu v_x = \frac{y^2}{2} C + \frac{y}{h}\left(\mu U - \frac{h^2}{2} C\right). \end{array}\right.$$



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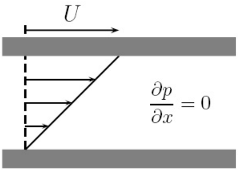
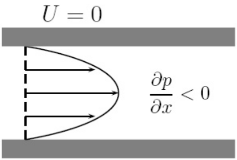


Couette flow

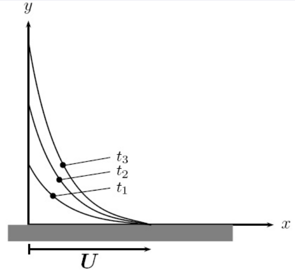


Poiseuille flow

Special cases

Rayleigh problem

- Instationary flow along plate
- Initially plate is at rest, then it starts to move at constant speed U
- No pressure gradient $\frac{\partial p}{\partial x} = 0$; $v_y = 0$; $v_x = v_x(y, t)$.



$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}, \qquad \begin{cases} t = 0: & v_x = 0, \\ y = 0: & v_x = U, \\ y \to \infty: & v_x \to 0. \end{cases}$$

Special cases

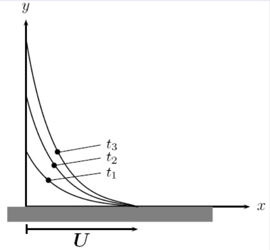
Rayleigh problem

- Introduce: $\xi = \frac{y}{\sqrt{4\nu t}}, \quad f(\xi) = \frac{v_x}{U}$.

$$f'' + 2\xi f' = 0, \qquad \begin{cases} \xi = 0: & f = 0, \\ \xi \to \infty: & f \to 0. \end{cases}$$

- Similarity solution:

$$f(\xi) = 1 - \mathrm{erf}(\xi)$$
$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-x^2}\, \mathrm{d}x.$$



Special cases

Perfect fluid

- A perfect fluid is non-viscous and satisfies the Euler equation

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\vec{\nabla} p + \rho \vec{b}\,.$$

- At the boundary the normal velocity component is zero.

Kelvin's theorem

In barotropic flow under conservative body forces, the velocity circulation Γ around any closed material contour is independent of time.

$$\Gamma = \oint_{\partial A} \vec{v} \cdot \mathrm{d}s = \int_A (\vec{\nabla} \times \vec{v}) \cdot \vec{n}\, \mathrm{d}S.$$

$\Rightarrow$ Irrotational flows remain irrotational.

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Bernoulli equation for steady incompressible flow

In steady, incompressible, barotropic flow under conservative body forces $\vec{b} = -\vec{\nabla}\Omega$,

$$p + \frac{\rho}{2} V^2 + \rho\Omega = \text{constant along streamlines} \quad (V = |\vec{v}|).$$

For irrotational flow it is constant everywhere.

Special cases

Acoustic waves of small amplitudes (1D)

- Neglecting body forces the 1D Euler and continuity equations are:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0.$$

- Rescale velocity with speed of sound c:

$$Sr_c \rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} + E_c \frac{\partial p}{\partial x} = 0, \qquad Sr_c \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0.$$

- Barotropic equation of state: $\rho = \rho(p)$.

Special cases

Acoustic waves of small amplitudes (1D)

- We linearize assuming small Mach number M:

$$v = Mv_1(x,t), \quad p = p_0 + Mp_1(x,t), \quad \rho = \rho_0 + M\rho_1(x,t).$$

- Approximate p_1 by $p_1 = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_0 \rho_1 =: \alpha\rho_1$ then

$$Sr_c\rho\frac{\partial v_1}{\partial t} + M\rho v_1\frac{\partial v_1}{\partial x} + \alpha E_c\frac{\partial \rho_1}{\partial x} = 0, \quad Sr_c\frac{\partial \rho_1}{\partial t} + \frac{\partial(\rho_0 v_1)}{\partial x} = 0.$$

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- v_1, p_1 and ρ_1 all satisfy the wave equation:

$$\frac{\partial^2\Psi}{\partial t^2} = c_0^2\frac{\partial^2\Psi}{\partial x^2}, \qquad c_0^2 = \frac{\alpha E_c}{Sr_c}.$$

- Solution is sum of right and left running wave:

$$\Psi(x,t) = f(x - c_0 t) + g(x - c_0 t).$$

Potential flow of incompressible perfect fluids

Potential flow

- We consider irrotational, incompressible, perfect fluids.
- Irrotational flow: $\vec{\nabla} \times \vec{v} = 0 \quad \longrightarrow \quad \vec{v} = \vec{\nabla}\phi.$
- Incompressibility: $\vec{\nabla} \cdot \vec{v} = \Delta\phi = 0.$

Potential flow in 2D

- Continuity equation: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.$

  $\Rightarrow$ there exists a function ψ, the stream function, such that

$$v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}.$$

- ψ is constant along streamlines.

Complex-function formulation

Holomorphic functions

A complex function $f(z) = u(x, y) + iv(x, y)$ is holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

and u and v have continuous first partial derivatives.

$\Rightarrow$ u and v both satisfy Laplace's equation.

$\Rightarrow$ Holomorphic functions are analytic functions.

Plane potential flow

Complex-function formulation

The complex potential function $f(z) = \phi(x) + i\psi(x)$ is holomorphic as ϕ and ψ satisfy Cauchy-Riemann.

Inverse method

Examine various holomorphic complex potential functions and choose one that's useful

Examples

- Power series of z within their circle of convergence.
- Logarithmic potentials away from their points of singularity.

$$\Rightarrow \quad f(z) = az, \qquad f(z) = m \ln(z - z_0).$$

Plane potential flow

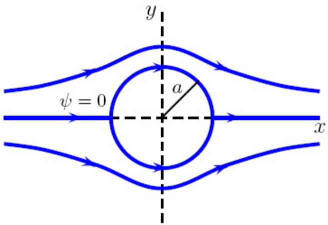
Symmetric flow past a circular cylinder

For real V, the complex potential

$$f(z) = V\left(z + \frac{a^2}{z}\right) = V\left(re^{\mathrm{i}\theta} + \frac{a^2}{r}e^{-\mathrm{i}\theta}\right),$$

has real and imaginary parts

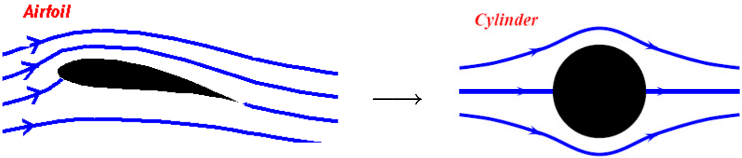
$$\phi = V\left(r + \frac{a^2}{r}\right)\cos\theta, \quad \psi = V\left(r - \frac{a^2}{r}\right)\sin\theta.$$



Plane potential flow

Conformal mappings

- Conformal mappings can be used to map simply connected subsets of the complex plane onto each other.
  - ⇒ A potential problem on a very complicated domain can be transformed to a potential problem on a simple domain.
  - ⇒ Flow over airfoils can be mapped conformally onto the exterior of a circular cylinder ([3]).



Summary

Fluid mechanics: Newtonian fluids

- General theorems:
  - $\Rightarrow$ equations of motion,
  - $\Rightarrow$ continuity equation,
  - $\Rightarrow$ energy equation.
- Constitutive equations.
- Simplifying assumptions:
  - $\Rightarrow$ steady flow,
  - $\Rightarrow$ incompressible flow,
  - $\Rightarrow$ irrotational flow,
  - $\Rightarrow$ Stokes condition,
  - $\Rightarrow$ barotropic fluids,
  - $\Rightarrow$ perfect (inviscid) fluids.

Further reading

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