

Principles of virtual work

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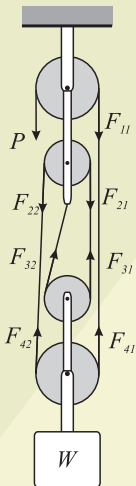
Formulation of principle of virtual work

Sum of works of the internal and external forces done by virtual displacements is zero

$$-\delta W_i - \delta W_e = 0.$$

Virtual displacements are infinitesimal change in the position coordinates of a system such that the constraints remain satisfied.

Example standard way

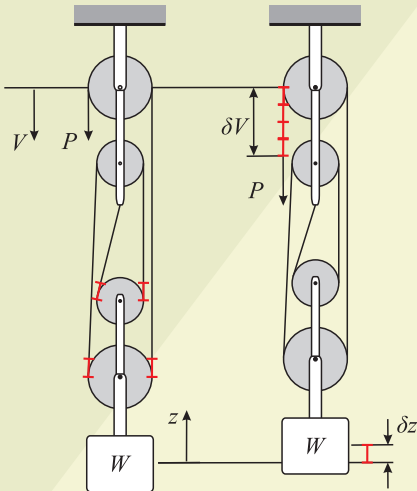


To find a relation between W and P we need to solve the system of **8** equations

$$\left\{ \begin{array}{l} P = F_{11} \\ F_{22} = F_{21} \\ F_{32} = F_{31} \\ F_{42} = F_{41} \\ F_{11} = F_{41} \\ F_{42} = F_{22} \\ F_{21} = F_{31} \\ F_{31} + F_{41} + F_{32} + F_{42} = W \end{array} \right.$$

and get $W = 4P$.

Example principle of virtual work



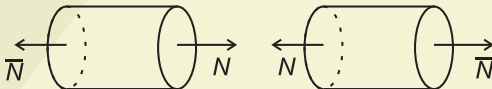
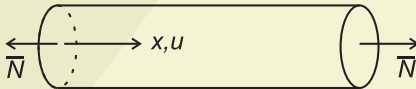
From the principle of virtual work,

$$-\delta W_i - \delta W_e = W\delta z - P\delta V = 0.$$

Geometrically admissible virtual displacements δV and δz are related according to $\delta V = 4\delta z$. Thus,

$$W = 4P.$$

Work & energy. Elastic extension of a bar



$$\bar{N} = -N = - \int_A \sigma_x dA = -\sigma_x A$$



Work & energy. Elastic extension of a bar

Internal work done in a differential element dx is

$$-\sigma_x d\epsilon_x A dx = -\sigma_x d\epsilon_x dV$$

with the strain $d\epsilon_x = d(du/dx)$ and the volume element $dV = A dx$.

The total work of the bar is

$$W_i = - \int_0^L \int_0^{\epsilon_x} \sigma_x d\epsilon_x A dx.$$

Using the Hooke's law $\sigma_x = E\epsilon_x$ we obtain

$$W_i = - \int_0^L A \int_0^{\epsilon_x} E\epsilon_x d\epsilon_x dx = -\frac{1}{2} \int_0^L AE\epsilon_x^2 dx.$$

Work of the internal forces in 3D

Consider a parallelepiped ($dx dy dz$). Then the work during the strain increment $d\epsilon_x, d\epsilon_y, \dots, d\gamma_{yz}$ would be

$$(\sigma_x d\epsilon_x + \sigma_y d\epsilon_y + \dots + \tau_{yz} d\gamma_{yz}) dx dy dz.$$

The stress tensor σ_{ij} and the strain tensor ϵ_{ij} are

$$\sigma_{ij} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix}, \quad \epsilon_{ij} = \begin{pmatrix} \epsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & \epsilon_y & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & \epsilon_z \end{pmatrix}.$$

Work of the internal forces in 3D

Then the work for the whole body of volume V becomes

$$\begin{aligned}
 W_i &= - \int_V \int_{(0,0,\dots,0)}^{(\epsilon_x, \epsilon_y, \dots, \gamma_{yz})} (\sigma_x d\epsilon_x + \sigma_y d\epsilon_y + \dots + \tau_{yz} d\gamma_{yz}) dx dy dz = \\
 &= - \int_V \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} dV = - \int_V \int_0^{\epsilon} \sigma^T d\epsilon dV,
 \end{aligned}$$

where $\sigma = [\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}]^T$ and
 $\epsilon = [\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}]^T$.

Using the Hooke's law $\sigma_{ij} = E_{ijkl}\epsilon_{kl}$ we arrive at

$$\begin{aligned} W_i &= -\frac{1}{2} \int_V E_{ijkl}\epsilon_{ij}\epsilon_{kl} dV = -\frac{1}{2} \int_V \epsilon^T \mathbf{E} \epsilon dV = \\ &= -\frac{1}{2} \int_V \sigma_{ij}\epsilon_{ij} dV = -\frac{1}{2} \int_V \sigma^T \epsilon dV. \end{aligned}$$

Next we introduce the strain energy density as

$$\begin{aligned} U_0(\epsilon) &= \frac{1}{2} \epsilon^T \mathbf{E} \epsilon = G \left(\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 + \right. \\ &\quad \left. \frac{\nu}{1-2\nu} (\epsilon_{11} + \epsilon_{22} + \epsilon_{33})^2 + \frac{1}{2} (\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2) \right) \end{aligned}$$

which is a positive definite function. The total strain energy is

$$U_i = -W_i = \int_V U_0(\epsilon) dV.$$

Complementarity strain energy density

Complementarity strain energy density is

$$U_0^* = \sigma_{ij}\epsilon_{ij} - U_0(\epsilon).$$

In case of Hooke's law

$$U_0^*(\sigma) = \sigma^T \epsilon - \frac{1}{2} \sigma^T \epsilon = \frac{1}{2} \sigma^T \mathbf{E}^{-1} \sigma = \frac{1}{2E} \left((\sigma_{11} + \sigma_{22} + \sigma_{33})^2 + (\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{22}\sigma_{33}) \right)$$

With a linear material law

$$U_0 = \frac{1}{2} \epsilon^T \mathbf{E} \epsilon = \frac{1}{2} \sigma^T \mathbf{E}^{-1} \sigma = U_0^*$$

The complementarity strain energy is

$$U_i^* = \int_V U_0^*(\sigma) dV = \frac{1}{2} \int_V \sigma^T \mathbf{E}^{-1} \sigma dV.$$

The complementarity work of the internal forces is given by

$$W_i^* = - \int_V \int^{\sigma_{ij}} \epsilon_{ij} d\sigma_{ij} dV = - \int_V \int^{\sigma} \epsilon d\sigma dV.$$

Extension of a bar

The external work performed until the final configuration u_F is reached

$$W_e = \int_0^{u_F} \bar{N} du.$$

If the displacement is proportional to the load $u = k\bar{N}$ we have

$$W_e = \int_0^{u_F} \bar{N} du = \int_0^{u_F} \frac{u}{k} du = \frac{1}{2} \frac{u_F^2}{k} = \frac{1}{2} \bar{N}_F u_F.$$

External work

The external work done by the prescribed forces \bar{p}_i , \bar{p}_{V_i} is

$$W_e = \int_{S_p} \int_0^{u_i} \bar{p}_i du_i dS + \int_V \int_0^{u_i} \bar{p}_{V_i} du_i dV.$$

For linear material

$$W_e = \frac{1}{2} \int_{S_p} \bar{p}_i u_i dS + \frac{1}{2} \int_V \bar{p}_{V_i} u_i dV.$$

Complementarity external work

The complementarity external work is

$$W_e^* = \int_{S_u} \int_0^{p_i} \bar{u}_i dp_i dS$$

For linear material

$$W_e^* = \frac{1}{2} \int_{S_u} p_i \bar{u}_i dS.$$

Energy for loading

Energy of external forces

$$U_e = - \int_{S_p} \int_0^{u_i} \bar{p}_i du_i dS - \int_V \int_0^{u_i} \bar{p}_{V_i} du_i dV.$$

Complementarity energy of external forces

$$U_e^* = - \int_{S_u} \int_0^{u_i} p_i \bar{u}_i dS.$$



Internal&external virtual work

The internal virtual work is given by

$$\delta W_i = - \int_V \sigma_{ij} \delta \epsilon_{ij} dV,$$

and the external virtual work is

$$\delta W_e = \int_{S_p} \bar{p}_i \delta u_i dS + \int_V \bar{p}_{V_i} \delta u_i dV,$$

where δu_i are the virtual displacements.



Internal&external complementarity virtual work

The internal complementarity virtual work is given by

$$\delta W_i^* = - \int_V \epsilon_{ij} \delta \sigma_{ij} dV,$$

and the external virtual work is

$$\delta W_e^* = \int_{S_u} \bar{u}_i \delta p_i dS.$$

Principle of virtual work (1)

The principle of virtual work can be derived from the equations of equilibrium and vice versa. The condition of equilibrium for a solid body under the prescribed body forces is

$$\sigma_{ij,j} + \bar{p}_{Vi} = 0 \text{ in } V. \quad (1)$$

The force boundary conditions are given on the part of the surface S_p

$$p_i = \bar{p}_i \text{ on } S_p \quad (2)$$

where $p_i = \sigma_{ij}a_j$. On the other part of the boundary S_u ($S = S_u \cup S_p$) displacements are prescribed.

Principle of virtual work (2)

Next we multiply (1) and (2) by the virtual displacement δu_i and integrate the first relation over V and the second one over S_p .

$$- \int_V (\sigma_{ij,j} + \bar{p}_{V_i}) \delta u_i dV + \int_{S_p} (p_i - \bar{p}_i) \delta u_i dS = 0. \quad (3)$$

The virtual displacements are kinematically admissible if the displacements $\hat{u}_i = u_i + \delta u_i$ satisfy the geometric boundary conditions ($u_i = \hat{u}_i$ on S_u),

$$\delta u_i = 0 \text{ on } S_u.$$

Principle of virtual work (3)

We use the Gauss integral theorem to receive

$$\int_V \sigma_{ij,j} \delta u_i dV = \int_S p_i \delta u_i dS - \int_V \sigma_{ij} \delta u_{i,j} dV. \quad (4)$$

From the definition of S_p we have

$$\int_{S_p} p_i \delta u_i dS = \int_S p_i \delta u_i dS - \int_{S_u} p_i \delta u_i dS. \quad (5)$$

We substitute (4) and (5) into (3)

$$\int_V \sigma_{ij} \delta \epsilon_{ij} dV - \int_V \bar{p}_{V_i} \delta u_i dV - \int_{S_p} \bar{p}_i \delta u_i dS - \int_{S_u} p_i \delta u_i dS = 0, \quad (6)$$

where $\delta \epsilon_{ij} = 1/2 \delta(u_{i,j} + u_{j,i})$.

Principle of virtual work (4)

Because the variations are kinematically admissible $\delta u_i = 0$ on S_u which gives us

$$\int_V \sigma_{ij} \delta \epsilon_{ij} dV - \int_V \bar{p}_{V_i} \delta u_i dV - \int_{S_p} \bar{p}_i \delta u_i dS = 0 \quad (7)$$

or in terms of internal and external virtual work

$$-\delta W_i - \delta W_e = 0. \quad (8)$$

(7) or (8) are called the principle of virtual work. This principle is also known as the principle of virtual displacements.

Principle of complementarity virtual work (1)

We start from the local kinematic conditions

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } V \quad (9)$$

and

$$u_i = \bar{u}_i \text{ on } S_u. \quad (10)$$

Multiply (9) by a virtual stress field $\delta\sigma_{ij}$ and integrate over the volume. Multiply (10) by the virtual force δp_i and integrate over the surface S_u . The sum of two gives

$$\int_V (\epsilon_{ij} - u_{i,j}) \delta\sigma_{ij} dV - \int_{S_u} (\bar{u}_i - u_i) \delta p_i dS = 0. \quad (11)$$

Principle of complementarity virtual work (2)

Here we introduced the concept of statically admissible stresses $\hat{\sigma}_{ij} = \sigma_{ij} + \delta\sigma_{ij}$ which satisfy the equilibrium equations in V and static boundary conditions on S_p . Then, it follows that

$$\delta\sigma_{ij,j} = 0 \text{ in } V, \quad (12)$$

$$\delta p_i = 0 \text{ on } S_p. \quad (13)$$

Using the Gauss integral theorem, (12), (13) and $\delta p_j = a_i \delta\sigma_{ij}$ we get

$$\int_{S_u} u_i \delta p_i dS = \int_V u_{i,j} \delta\sigma_{ij} dV.$$

Principle of complementarity virtual work (3)

By combining (11) and (8) we receive

$$\int_V \epsilon_{ij} \delta \sigma_{ij} dV - \int_{S_u} \bar{u}_i \delta p_i dS = 0,$$

or in terms of external and internal complementary virtual work

$$-\delta W_i^* - \delta W_e^* = 0.$$

(7) or (8) are called the principle of complementary virtual work. This is also known as the principle of virtual stresses and as the principle of virtual forces.

Conclusions

- Work and potential energy of the internal forces
- Work and potential energy of the applied loading
- Principle of virtual work
- Principle of complementary virtual work