Reconstruction, Approximation and the ENO - Schemes

CASA Seminar

Clemens A. Zarzer

4th October 2006

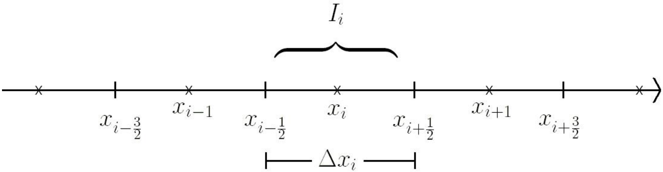
Outline

1. Motivation
2. Reconstruction
3. Approximation
4. ENO Reconstruction

1D Conservation Law

$$\mathrm{u_t(x,t) + f_x(u(x,t)) = 0 \quad x \in \Omega}$$

- Domain Ω and one dimensional “cell-structure”:



- Considered example:

$$\mathrm{f(u(x,t)) = \frac{u(x,t)^2}{2}} \quad \ldots \text{ Burgers' equation .}$$

Finite Volume Method

- Balance form for cell I_i:

$$\int_{I_i} u_t(\xi,t)\,d\xi + f(u(x_{i+\frac{1}{2}},t)) - f(u(x_{i-\frac{1}{2}},t)) = 0 \; .$$

- Idea: Approximate flux f by a numerical flux $\hat{f}$:

$$\frac{d}{dt}\bar{u}_i(t) = -\frac{1}{\Delta x_i}\left(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}\right) \; ,$$

where

$$\bar{u}_i(t) = \frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(\xi,t)\,d\xi \; .$$

Numerical Flux

- Numerical flux-functions $\mathrm{h(u, v)}$ are the Godunov, the Engquist-Osher or the Lax-Friedrichs flux-function, etc.
- Approximation: $\hat{\mathrm{f}}_{\mathrm{i}+\frac{1}{2}} = \mathrm{h}(\mathrm{u}^-_{\mathrm{i}+\frac{1}{2}}, \mathrm{u}^+_{\mathrm{i}+\frac{1}{2}})$.
- Hence $\mathrm{u}^-_{\mathrm{i}+\frac{1}{2}}$ and $\mathrm{u}^+_{\mathrm{i}+\frac{1}{2}}$ have to be reconstructed.

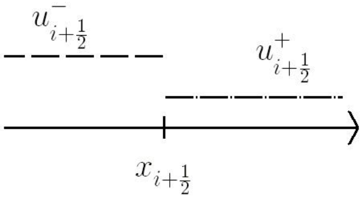


Figure: sketch of cell boundary

Numerical Flux

- Numerical flux-functions $\mathrm{h}(\mathrm{u}, \mathrm{v})$ are the Godunov, the Engquist-Osher or the Lax-Friedrichs flux-function, etc.
- Approximation: $\hat{\mathrm{f}}_{\mathrm{i}+\frac{1}{2}} = \mathrm{h}(\mathrm{u}^-_{\mathrm{i}+\frac{1}{2}}, \mathrm{u}^+_{\mathrm{i}+\frac{1}{2}})$.
- Hence $\mathrm{u}^-_{\mathrm{i}+\frac{1}{2}}$ and $\mathrm{u}^+_{\mathrm{i}+\frac{1}{2}}$ have to be reconstructed.

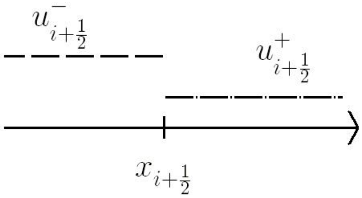


Figure: sketch of cell boundary

Problem Specification

one-dimensional reconstruction:

For given cell averages of the function v(x),

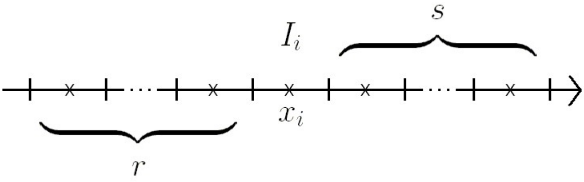
$$\bar{v}_i := \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} v(\xi)\, d\xi \quad , \; i \in \{1, \ldots, N\}$$

find a k-th order accurate estimates for the values of v(x) at the cell boundaries.

- Particularly the basis of this approximation is formed by polynomials of degree at most $k - 1$.

Polynomial Setup

- For each cell $\mathrm{I_i}$ a particular stencil is chosen.



$$r + s + 1 = k$$

Figure: k-th order “stencil” for cell $\mathrm{I_i}$

- Let $\mathrm{S(i) = \{I_{i-r}, \ldots, I_{i+s}\}}$ - the cell stencil.
- Let $\mathrm{\tilde{S}(i) = \{x_{i-r-\frac{1}{2}}, \ldots, x_{i+s+\frac{1}{2}}\}}$ - the point stencil.

Interpolating Polynomial

- Based on the average values in the stencil a unique polynomial is determined.
- The unique polynomial for cell $\mathrm{I_i}$ is determined via

$$\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p_i(\xi)\, d\xi = \bar{v}_j \;, \quad I_j \in S(i) \;.$$

- One easily proves the desired property,

$$p_i(x) = v(x) + O({\Delta x_i}^k), \; x \in I_i \;.$$

Efficient Approach

- Let $v_{i+\frac{1}{2}} = p_i(x_{i+\frac{1}{2}})$.
- The values at the cell boundaries, $v_{i+\frac{1}{2}}$, depend linearly on the given average values $\bar{v}_j$.

Hence there exist constants c_{rj}, such that:

$$v_{i+\frac{1}{2}} = \sum_{j=0}^{k-1} c_{rj}\, \bar{v}_{i-r+j} \ ,$$

where

$$v_{i+\frac{1}{2}} = v(x_{i+\frac{1}{2}}) + O({\Delta x_i}^k) \ .$$

Calculation of the Constants

- Consider $V(x)$ as the primitive function of $v(x)$.
- $V(x_{i+\frac{1}{2}})$ can be expressed as

$$V(x_{i+\frac{1}{2}}) = \sum_{j=-\infty}^{i} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v(\xi)\,d\xi = \sum_{j=-\infty}^{i} \bar{v}_j\,\Delta x_j\ .$$

- Consequently one can compute the unique interpolating polynomial $P_i(x)$.
- One easily proves: $P_i'(x) = p_i(x)$, where

$$\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p_i(\xi)\,d\xi = \bar{v}_j\ , \quad I_j \in S(i)\ .$$

Calculation of the Constants

$$\mathrm{P_i(x)\ interpolates\ V(x)} \qquad \wedge \qquad \mathrm{P_i(x)' = p_i(x)}$$

- Lagrange form of $\mathrm{P_i(x)}$

$$\mathrm{P_i(x)} = \sum_{\mathrm{m=0}}^{\mathrm{k}} \mathrm{V(x_{i-r-m-\frac{1}{2}})} \prod_{\substack{\mathrm{l=0} \\ \mathrm{l\neq m}}}^{\mathrm{k}} \frac{\mathrm{x - x_{i-r+l-\frac{1}{2}}}}{\mathrm{x_{i-r+m-\frac{1}{2}} - x_{i-r+l-\frac{1}{2}}}}$$

- Taking the derivative of $\mathrm{P_i(x)}$ and using that $\mathrm{V(x_{i+\frac{1}{2}})}$ can be expressed by the average values, gives

$$\mathrm{p_i(x)} = \sum_{\mathrm{j=0}}^{\mathrm{k-1}} \alpha_{\mathrm{rj}}^{\mathrm{(i)}} \, \Delta \mathrm{x_{i-r+j}} \, \mathrm{\bar{v}_{i-r+j}} \ .$$

Approximation

one-dimensional conservative approximation:

- Available data:
  a set of point values of a function v(x)
- Aim:
  find a numerical flux $\hat{v}$, which approximates $v'(x)$:

$$\frac{1}{\Delta x_i}\left(\hat{v}_{i+\frac{1}{2}} - \hat{v}_{i-\frac{1}{2}}\right) = v'(x_i) + O({\Delta x_i}^k)\ ,$$

$I_i \in S(i)$.

Solution Concept

Find a function h(x), such that

$$v(x) = \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} h(\xi)\, d\xi \ ,$$

then $v'(x) = \frac{1}{\Delta x}\left[h\left(x+\frac{\Delta x}{2}\right) - h\left(x-\frac{\Delta x}{2}\right)\right]$.

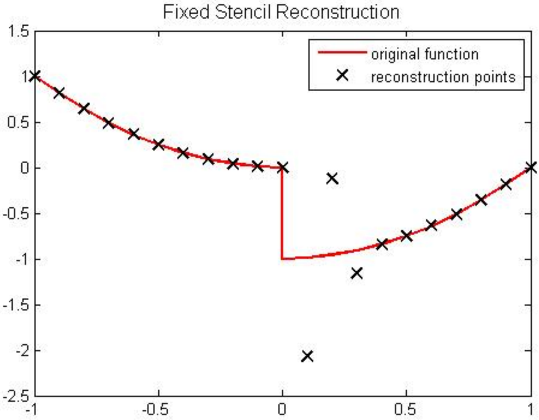
point values of v(x) $\hat{=}$ average values of h(x)

$\Downarrow$

reconstruction based on average values

Discontinuous Functions

- Classical methods can not be applied to discontinuous functions in general.



Main Idea

choose stencil for each cell with respect to the discontinuity

$$\Downarrow$$

ADAPTIVE STENCIL

- Particularly the left shift r is chosen such that, the "discontinuous cell" is not included - if possible.
- This requires a calculable quantity, which indicates the "smoothness" of the function.

Newton Divided Differences

The Newton polynomial of degree k, interpolating V(x):

$$\sum_{j=0}^{k} V[x_{i-r-\frac{1}{2}}, \dots, x_{i-r+j-\frac{1}{2}}] \prod_{m=0}^{j-1} (x - x_{i-r+m-\frac{1}{2}}) \,,$$

where

$$V[x_{i-\frac{1}{2}}, \dots, x_{i+j-\frac{1}{2}}] = \frac{V[x_{i+\frac{1}{2}}, \dots, x_{i+j-\frac{1}{2}}] - V[x_{i-\frac{1}{2}}, \dots, x_{i+j-\frac{3}{2}}]}{x_{i+j-\frac{1}{2}} - x_{i-\frac{1}{2}}}$$

are the so-called Newton Divided Differences.

ENO Idea

- Newton Divided Differences indicate smoothness

$$\begin{aligned} V[x_{i-\frac{1}{2}},\ldots,x_{i+j-\frac{1}{2}}] &= \frac{V^{(j)}(\xi)}{j!}, \quad \text{for smooth functions} \\ V[x_{i-\frac{1}{2}},\ldots,x_{i+j-\frac{1}{2}}] &= O\left(\frac{1}{\Delta x^j}\right), \quad \begin{array}{c}\text{in the case}\\ \text{of discontinuities}\end{array} \end{aligned}$$

Idea:

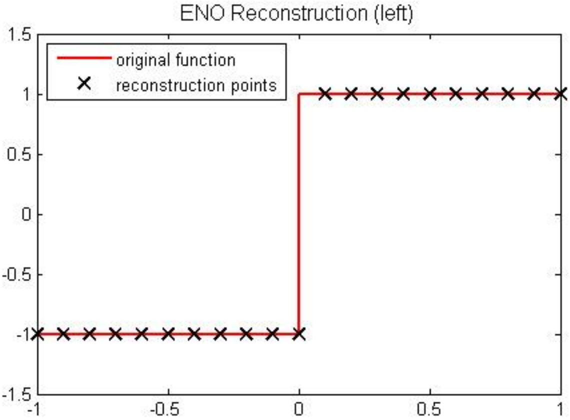
- start with a two point stencil
- add in each step another cell depending on the value of the Newton Divided Differences

One-Dimensional ENO Reconstruction

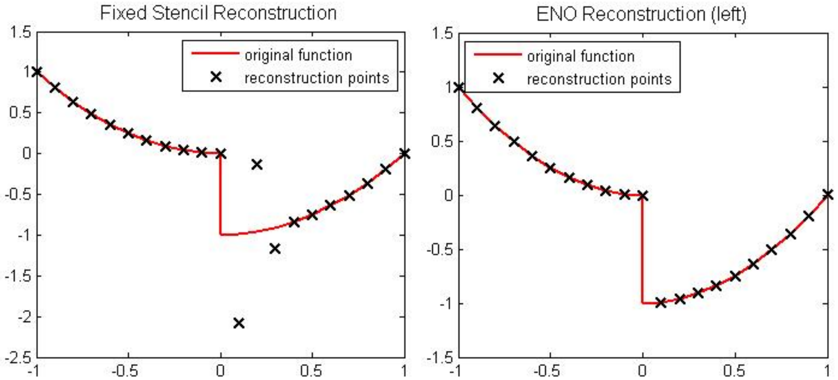
ENO procedure:

1. Compute the divided differences, using the cell averages.
2. For cell I_i start with the two-point stencil $\tilde{S}(i)$.
3. Assuming the actual stencil to be $\{x_{j+\frac{1}{2}}, \dots, x_{j+l-\frac{1}{2}}\}$, add
   - $x_{j-\frac{1}{2}}$ if $|V[x_{j-\frac{1}{2}}, \dots, x_{j+l-\frac{1}{2}}]| < |V[x_{j+\frac{1}{2}}, \dots, x_{j+l+\frac{1}{2}}]|$
   - $x_{j+l+\frac{1}{2}}$ otherwise
4. Repeat step 3. unless a k-th order stencil is obtained.
5. Use reconstruction method to compute the approximation to $v(x)$.

Numerical Results (1)



Numerical Results (2)



Burgers' Equation