The WENO-Schemes and 1D applications

CASA Seminar

Erwin Karer

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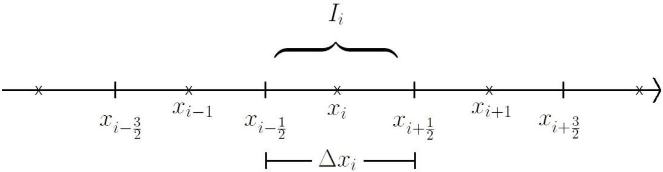
Outline

1. Motivation
2. Reconstruction
3. WENO
4. Numerical Implementation & Results

1D Conservation Law

$$u_t(x,t) + f_x(u(x,t)) = 0 \quad x \in \Omega$$

- Domain Ω and one dimensional “cell-structure”:



- Considered example:

$$f(u(x,t)) = \frac{u(x,t)^2}{2} \quad \ldots \text{ Burgers' equation .}$$

Finite Volume Method

- Balance form for cell I_i:

$$\int_{I_i} u_t(\xi, t)\, d\xi + f(u(x_{i+\frac{1}{2}}, t)) - f(u(x_{i-\frac{1}{2}}, t)) = 0 \,.$$

- Idea: Approximate flux f by a numerical flux $\hat{f}$:

$$\frac{d}{dt}\bar{u}_i(t) = -\frac{1}{\Delta x_i}\left(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}\right) \,,$$

where

$$\bar{u}_i(t) = \frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(\xi, t)\, d\xi \,.$$

Numerical Flux

- Numerical flux-functions $\mathrm{h}(\mathrm{u},\mathrm{v})$ are the Godunov, the Engquist-Osher or the Lax-Friedrichs flux-function, etc.
- Approximation: $\hat{\mathrm{f}}_{\mathrm{i}+\frac{1}{2}} = \mathrm{h}(\mathrm{u}^{-}_{\mathrm{i}+\frac{1}{2}}, \mathrm{u}^{+}_{\mathrm{i}+\frac{1}{2}})$.
- Hence $\mathrm{u}^{-}_{\mathrm{i}+\frac{1}{2}}$ and $\mathrm{u}^{+}_{\mathrm{i}+\frac{1}{2}}$ have to be reconstructed.

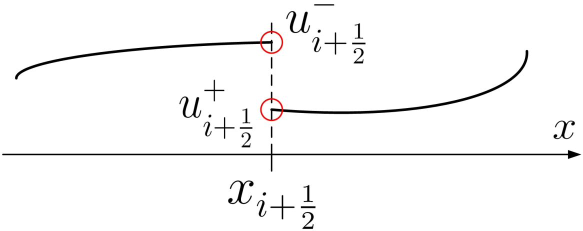


Figure: sketch of cell boundary

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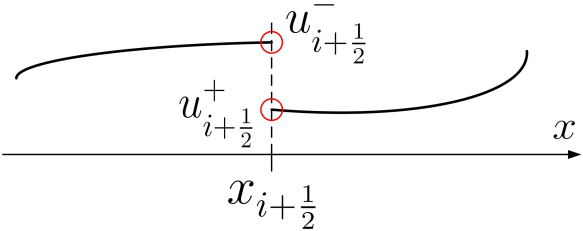


Figure: sketch of cell boundary

The Task of Reconstruction

Given are the cell averages $\{\bar{\mathrm{v}}_i\}_{i=1,\dots,N}$ with

$$\bar{\mathrm{v}}_i := \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathrm{v}(\xi)\, d\xi \ \ , \ i \in \{1, \dots, N\}$$

We want:

- to reconstruct the cell boundary values $\mathrm{v}_{i+\frac{1}{2}}$ and $\mathrm{v}_{i-\frac{1}{2}}$.
- to achieve a certain accuracy k, i.e.

$$\mathrm{v}_{i\pm\frac{1}{2}} = \mathrm{v}(x_{i\pm\frac{1}{2}}) + \mathcal{O}(\Delta x^k) \quad i = 1, \dots, N.$$

ENO Idea

For given order k and fixed left-shift r one is able to obtain a reconstruction of $v(x_{i+\frac{1}{2}})$:

$$v_{i+\frac{1}{2}} = \sum_{j=0}^{k-1} c_{rj}\, \bar{v}_{i-r+j} \ .$$

with special c_{rj}. (Fixed Stencil Approximation)

Newton Divided Differences indicate smoothness!
Idea of ENO:

- start with a two point stencil
- add in each step another cell depending on the value of the Newton Divided Differences

$$\Downarrow$$

Result is the left shift r!

Properties of ENO

k stencils are considered in the choosing process covering $2k-1$ cells. But only one stencil is used to reconstruct. One could try to get $(2k-1)$-th accuracy.

Further properties:

1. The stencil might change even by round-off error perturbation. If both Newton Divided Differences are near 0 a small change at the round off level would change the stencil.
2. The numerical flux is not smooth. The stencil pattern might change at neighboring cells.
3. ENO stencil choosing procedure contains many “if” directives which are not efficient on certain vector computers.

Basics

Assume a uniform grid, i.e. $\Delta x_i = \Delta x$, $i = 1, \ldots, N$. Let us define the k candidate stencils

$$S_r(i) = \{x_{i-r}, \ldots, x_{i-r+k-1}\} \qquad r = 0, \ldots, k-1.$$

The reconstruction produces k different approximations to the value $v(x_{i+\frac{1}{2}})$:

$$v^{(r)}_{i+\frac{1}{2}} = \sum_{j=0}^{k-1} c_{rj} \bar{v}_{i-r+j} \qquad r = 0, \ldots, k-1$$

WENO takes a convex combination of the $v^{(r)}_{i+\frac{1}{2}}$'s!

New Approximation

We get a new approximation to the value $\mathrm{v}(\mathrm{x}_{\mathrm{i}+\frac{1}{2}})$:

$$\mathrm{v}_{\mathrm{i}+\frac{1}{2}} = \sum_{\mathrm{r}=0}^{\mathrm{k}-1} \omega_{\mathrm{r}} \mathrm{v}^{(\mathrm{r})}_{\mathrm{i}+\frac{1}{2}}.$$

Due to consistency and stability we require

$$\begin{aligned} \omega_{\mathrm{r}} &\geq 0 \qquad \forall \mathrm{r} = 0, \dots, \mathrm{k}-1 \\ \sum_{\mathrm{r}=0}^{\mathrm{k}-1} \omega_{\mathrm{r}} &= 1. \end{aligned}$$

If function v is smooth on all candidate stencils there are constants d_{r} such that one obtains

$$\mathrm{v}(\mathrm{x}_{\mathrm{i}+\frac{1}{2}}) = \sum_{\mathrm{r}=0}^{\mathrm{k}-1} \mathrm{d}_{\mathrm{r}} \mathrm{v}^{(\mathrm{r})}_{\mathrm{i}+\frac{1}{2}} + \mathcal{O}(\Delta \mathrm{x}^{2\mathrm{k}-1}).$$

Properties

Due to constistency we have

$$\sum_{r=0}^{k-1} d_r = 1.$$

If v is smooth on all $S_r(i)$ we would like to have

$$\omega_r = d_r + \mathcal{O}(\Delta x^{k-1}),$$

which would imply the $(2k-1)$-th order accuracy

$$v_{i+\frac{1}{2}} = \sum_{r=0}^{k-1} \omega_r v_{i+\frac{1}{2}}^{(r)} = v(x_{i+\frac{1}{2}}) + \mathcal{O}(\Delta x^{2k-1})$$

Calculation of Coefficients d_r

Let us define the described reconstruction by

$$q_r^k(\bar{v}_{i-r}, \dots, \bar{v}_{i-r+k-1}) := \sum_{j=0}^{k-1} c_{rj}^k \bar{v}_{i-r+j}$$

Let us define the d_r via the relation

$$q_{k-1}^{2k-1}(\bar{v}_{i-k+1}, \dots, \bar{v}_{i+k-1}) = \sum_{r=0}^{k-1} d_r v_{i+\frac{1}{2}}^{(k-1-r)}.$$

Explanation of the Calculation of the $\mathrm{d_r}$

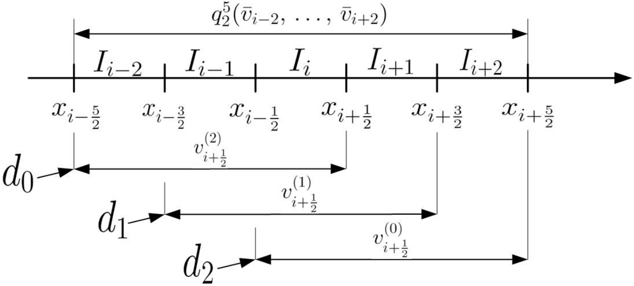


Figure: Calculation of the $\mathrm{d_r}$

Calculation of the $\mathrm{d_r}$:

- Take $\mathrm{k-1}$ sequential points $\mathrm{i} \Rightarrow \mathrm{k-1}$ equations
- The $\mathrm{d_r}$ have to fulfill $\sum_{\mathrm{r=0}}^{\mathrm{k-1}} \mathrm{d_r} = 1 \Rightarrow 1$ equation
- Solve the k equations for the k unknowns.

Coefficients ω_r

We require the following properties of the coefficients:

- If v(x) is smooth on $\mathrm{S_r(i)}$ we want
$$\omega_\mathrm{r} \approx \mathrm{d_r}$$
hence
$$\omega_\mathrm{r} = \mathcal{O}(1).$$
- If v(x) has a discontinuity inside $\mathrm{S_r(i)}$ ω_r should be essentially 0, i.e.
$$\omega_\mathrm{r} = \mathcal{O}(\Delta \mathrm{x}^\ell).$$

Calculation of ω_{r}

Combining these aspects we get

$$\omega_{\mathrm{r}} = \frac{\alpha_{\mathrm{r}}}{\sum_{\mathrm{s}=0}^{\mathrm{k}-1} \alpha_{\mathrm{s}}}, \qquad \mathrm{r} = 0, \ldots, \mathrm{k} - 1$$

where α_{r} is given by

$$\alpha_{\mathrm{r}} = \frac{\mathrm{d}_{\mathrm{r}}}{(\varepsilon + \beta_{\mathrm{r}})^2}.$$

with $\varepsilon \in [10^{-5}, 10^{-7}]$ (usually 10^-6). The β_{r} are the so-called "smooth indicators".

Smooth indicator β_r

If $v(x)$ is smooth in $S_r(i)$ one wants

$$\beta_r = \mathcal{O}(\Delta x^2)$$

and for discontinuous v one obtains

$$\beta_r = \mathcal{O}(1).$$

One choice for β_r which fulfills the requirements is

$$\beta_r = \sum_{l=1}^{k-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Delta x^{2l-1} \left(\frac{\partial^l p_r(x)}{\partial x^l} \right)^2 dx.$$

WENO Procedure

1. Calculate the k reconstructed values $v^{(r)}_{i+\frac{1}{2}}$ for each i.
2. Find the constants d_r.
3. Find the smooth indicators β_r.
4. Form the weights ω_r.
5. The $(2k-1)$-th order reconstruction is given by

$$v_{i+\frac{1}{2}} = \sum_{r=0}^{k-1} \omega_r v^{(r)}_{i+\frac{1}{2}}.$$

Note, that the values $v_{i+\frac{1}{2}}$ are later denoted by $v^{-}_{i+\frac{1}{2}}$. Additionally one can obtain the values $v^{+}_{i-\frac{1}{2}}$ for the cell i using the same considerations.

Comparison of the Reconstruction

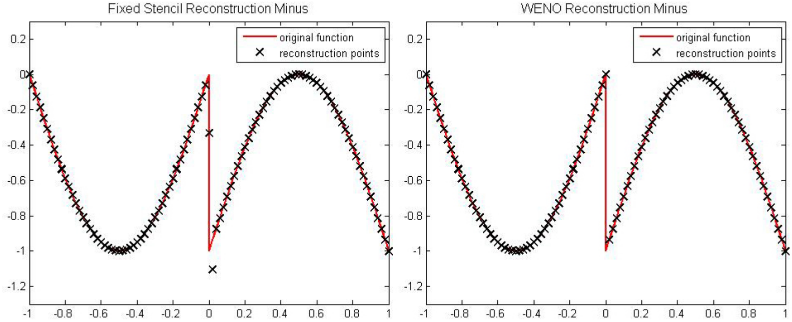
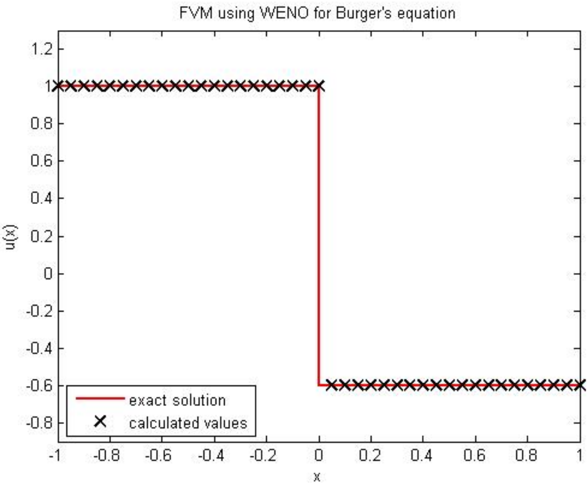


Figure: Comparison of the fixed-stencil approx.(fixed r = 1) left and the WENO(k = 3) right for $\Delta x = 0.02$.

Burger's equation using FVM and WENO



FDM approximation

We use a conservative approximation to the spatial derivative

$$\frac{du_i(t)}{dt} = -\frac{1}{\Delta x}\left(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}\right).$$

$u_i(t)$ is the numerical approximation to the point value $u(x_i, t)$. We want

$$\frac{1}{\Delta x_i}\left(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}\right) = f_x(u(x_i, t)) + \mathcal{O}(\Delta x^k) \qquad \forall i.$$

This numerical flux is obtained by ENO or WENO reconstruction using the setting

$$\bar{v}(x) = f(u(x, t)).$$

Upwinding using the Roe speed

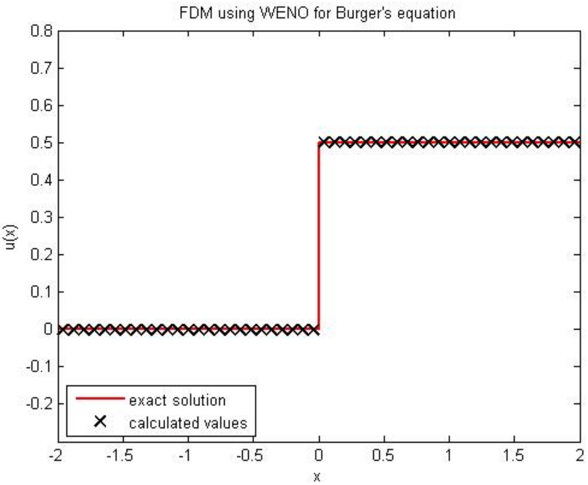
We have to define the Roe speed

$$\bar{a}_{i+\frac{1}{2}} := \frac{f(u_{i+1}) - f(u_i)}{u_{i+1} - u_i}.$$

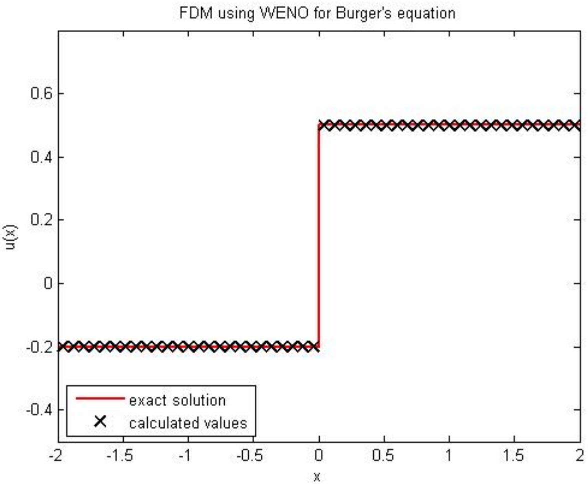
Depending on the sign of the $\bar{a}_{i+\frac{1}{2}}$ one uses $v^-_{i+\frac{1}{2}}$ or $v^+_{i+\frac{1}{2}}$ for the numerical flux, i.e.

- if $\bar{a}_{i+\frac{1}{2}} \geq 0$ one says that the wind blows from the left, hence one uses $v^-_{i+\frac{1}{2}}$ for the numerical flux $\hat{f}_{i+\frac{1}{2}}$.
- if $\bar{a}_{i+\frac{1}{2}} < 0$ one says that the wind blows from the right, hence one uses $v^+_{i+\frac{1}{2}}$ for the numerical flux $\hat{f}_{i+\frac{1}{2}}$.

Burger's equation using FDM and WENO-Roe



Burger's equation using FDM and WENO-Roe 2



Flux-splitting

A more stable approach is to use a splitting of the flux, i.e.

$$\mathrm{f}(\mathrm{u}) = \mathrm{f}^{+}(\mathrm{u}) + \mathrm{f}^{-}(\mathrm{u})$$

where

$$\frac{\mathrm{d}\mathrm{f}^{+}(\mathrm{u})}{\mathrm{d}\mathrm{u}} \geq 0 \quad \text{and} \quad \frac{\mathrm{d}\mathrm{f}^{-}(\mathrm{u})}{\mathrm{d}\mathrm{u}} \leq 0.$$

hold. For example the Lax-Friedrich splitting:

$$\mathrm{f}^{\pm}(\mathrm{u}) = \frac{1}{2}(\mathrm{f}(\mathrm{u}) \pm \alpha \mathrm{u})$$

where α is defined by

$$\alpha = \max_{\mathrm{u}} |\mathrm{f}'(\mathrm{u})|$$

Flux splitting procedure

The numerical flux is then obtained by the following procedure:

1. Identify $\bar{v}_i = f^+(u(x_i))$ and use ENO or WENO reconstruction procedure to obtain the values $v^-_{i+\frac{1}{2}}$.
2. Set the positive numerical flux as
$$\hat{f}^+_{i+\frac{1}{2}} = v^-_{i+\frac{1}{2}}.$$
3. Identify $\bar{v}_i = f^-(u(x_i))$ and use ENO or WENO reconstruction procedure to obtain the values $v^+_{i+\frac{1}{2}}$.
4. Set the negative numerical flux as
$$\hat{f}^-_{i+\frac{1}{2}} = v^+_{i+\frac{1}{2}}.$$
5. Form the numerical flux as
$$\hat{f}_{i+\frac{1}{2}} = \hat{f}^+_{i+\frac{1}{2}} + \hat{f}^-_{i+\frac{1}{2}}.$$

Burger's equation using FDM with flux-splitting