ENO and WENO Schemes for Hyperbolic Conservation Laws

Extension to Systems and Multi Dimensions

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CASA Seminar, 2006

Outline

1. Overview
2. Multi Space Dimensions
   - 2D Reconstruction for FV Schemes.
   - FV ENO/WENO Schemes for 2D Conservation Laws.
   - 2D Reconstruction for FD Schemes.
   - FD ENO/WENO Schemes for 2D Conservation Laws.
3. Systems of Conservation Laws
   - Component-wise Approach
   - Characteristic-wise Approach
4. Numerical Results
   - Dam-break Problem

Conservation Laws

1D scalar conservation law

$$u_t(x,y,t) + f_x(u(x,y,t)) = 0$$

$$+ICs + BCs$$

2D scalar conservation law

$$u_t(x,y,t) + f_x(u(x,y,t)) + g_y(u(x,y,t)) = 0$$

$$+ICs + BCs$$

System of conservation laws

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = 0 \quad + ICs + BCs$$

Solving 1D Scalar Conservation Laws Using ENO/WENO.

2 approaches:

- Finite Volume (FV) approach
  -> Reconstruction from cell averages of the conserved variables
- Finite Difference (FD) approach
  -> Reconstruction from point values of the flux

Finite Volume Approach.

Integrated version of the conservation law:

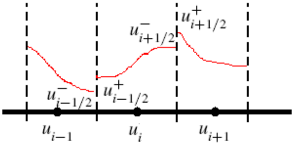
$$\frac{d\bar{u}_i(t)}{dt} = -\frac{1}{\Delta x_i}\left(f(u(x_{i+\frac{1}{2}},t)) - f(u(x_{i-\frac{1}{2}},y,t))\right)$$

Approximate the physical flux $f(u(x_{i+\frac{1}{2}},t))$ with a numerical flux $\hat{f}_{i+\frac{1}{2}}$

$$\hat{f}_{i+1/2} = h(u^-_{i+1/2}, u^+_{i+1/2})$$

h - monotone flux (Lipschitz continuous, $h(\uparrow,\downarrow)$, $h(a,a) = f(a)$) $\rightarrow$ TVD

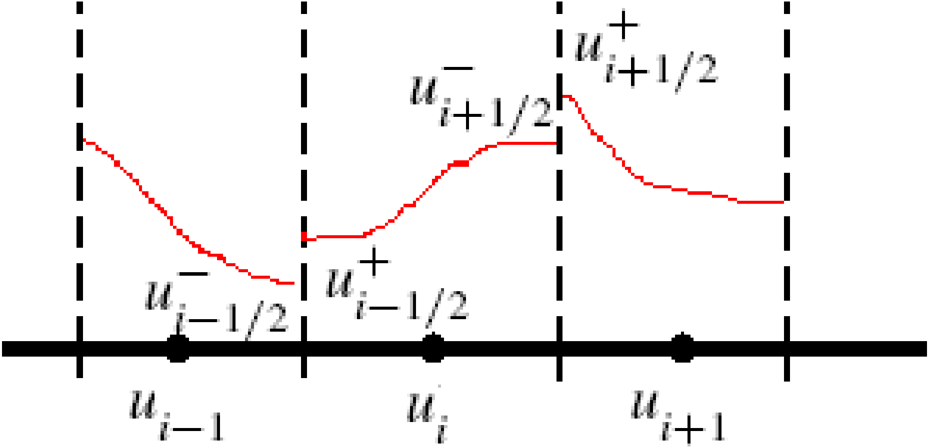
Example: $h(a,b) = 0.5(f(a) + f(b) - \alpha(b-a))$, where $\alpha = \max_u |f'(u)|$



Use ENO/WENO to compute $u^{\pm}_{i+1/2}$

$$u^-_{i+1/2} = p_i(x_{i+1/2}) = v_i(u_{i-r},\dots,u_{i+s})$$
$$u^+_{i+1/2} = p_{i+1}(x_{i+1/2}) = v_{i+1}(u_{i-r},\dots,u_{i+s})$$

Finite Volume Approach.



Finite Difference Approach.

Conservation law written using a conservative approximation to the spatial derivative:

$$\frac{du_i(t)}{dt} = -\frac{1}{\Delta x}(\hat{f}_{i+1/2} - \hat{f}_{i-1/2})$$

where $\hat{f}_{i+1/2}$ is the numerical flux $\hat{f}_{i+1/2} = \hat{f}(u_{i-r}, u_{i-r+1}, \ldots, u_{i+s})$
$\hat{f}_{i+1/2}$ is obtained by ENO/WENO procedure.
ENO/WENO => $\hat{f}^{+}_{i+1/2}$ and $\hat{f}^{-}_{i+1/2}$ => which one to use?
Compute Roe speed

$$a_{i+1/2} = \frac{f(u_{i+1}) - f(u_i)}{u_{i+1} - u_i}$$

If $a_{i+1/2} > 0$ use $\hat{f}^{-}_{i+1/2}$ (wind blows from left)
If $a_{i+1/2} \leq 0$ use $\hat{f}^{+}_{i+1/2}$ (wind blows from right)

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General Framework (1).

NOTE: Although we concetrate our attention on 2D procedures, things carry over to higher dimension as well.

We consider Cartesian grids. The domain is a rectangle

$$[a,b] \times [c,d]$$

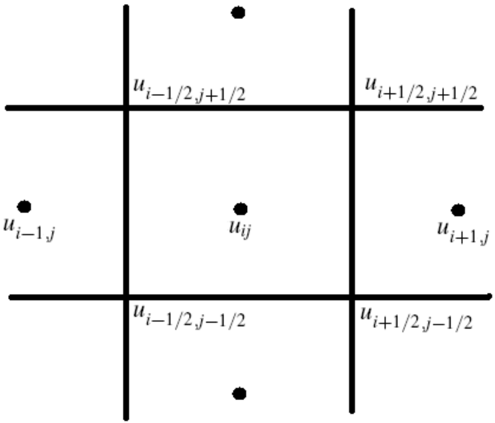
covered by cells

$$I_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{i-1/2}, y_{i+1/2}], \quad 1 \leq i \leq N_x, 1 \leq j \leq N_y$$

$$a = x_{1/2} \leq x_{3/2} \leq \ldots \leq x_{N_x-1/2} \leq x_{N_x+1/2} = b,$$

$$c = y_{1/2} \leq y_{3/2} \leq \ldots \leq y_{N_y-1/2} \leq y_{N_y+1/2} = d.$$

Two Dimensional Cell Array (figure).



General Framework (2).

The centers of the cells are

$$(x_i, y_j), \quad x_i = \frac{1}{2}(x_{i-1/2} + x_{i+1/2}), y_j = \frac{1}{2}(y_{j-1/2} + y_{j+1/2})$$

To denote the grid sizes we use

$$\Delta x_i \equiv x_{i+1/2} - x_{i-1/2}, \; i = 1, 2, ..., N_x$$

$$\Delta y_j \equiv y_{j+1/2} - y_{j-1/2}, \; j = 1, 2, ..., N_y$$

We denote the maximum grid size by

$$\Delta x \equiv \max_{1 \le i \le N_x} \Delta x_i, \;\; \Delta y \equiv \max_{1 \le j \le N_y} \Delta y_j$$

Finally

$$\Delta \equiv max(\Delta x, \Delta y)$$

Reconstruction from cell averages (1).

Problem formulation

Given the cell averages of a function $v(x, y)$:

$$\bar{v}_{ij} \equiv \frac{1}{\Delta x_i \Delta y_j} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} v(\xi, \eta)\, d\xi\, d\eta.$$

find a polynomial $p_{ij}(x, y)$ of degree $k-1$, for each cell I_{ij}, such that it is a k-th order accurate approximation to the function $v(x, y)$ inside I_{ij}:

$$p_{ij}(x, y) = v(x, y) + O(\Delta^k)$$

$$\text{for } (x, y) \in I_{ij},\ i = 1, 2, \dots, N_x,\ j = 1, 2, \dots, N_y.$$

We will use this polynomial to reconstruct the values at cell interface.

Reconstruction from cell averages (2).

This polynomial, evaluated at cell boundaries, gives the approximations

$$v^-_{i+1/2,y} = p_{ij}(x_{i+1/2}, y),\; v^+_{i-1/2,y} = p_{ij}(x_{i-1/2}, y)$$

$$i = 1, \ldots, N_x,\; y_{j-1/2} \leq y \leq y_{j+1/2}$$

$$v^-_{x,j+1/2} = p_{ij}(x, y_{j+1/2}),\; v^+_{x,j-1/2} = p_{ij}(x, y_{j-1/2})$$

$$j = 1, \ldots, N_y,\; x_{i-1/2} \leq x \leq x_{i+1/2}$$

which are k-th order accurate.

Reconstruction from cell averages (3).

If we use products of 1D polynomials:

$$p(x,y) = \sum_{m=0}^{k-1} \sum_{l=0}^{k-1} a_{lm} x^l y^m$$

then things can proceed as in 1D.
We introduce the 2D primitive:

$$V(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} v(\xi,\eta)\, d\xi\, d\eta.$$

Then

$$V(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) = \int_{-\infty}^{y_{j+\frac{1}{2}}} \int_{-\infty}^{x_{i+\frac{1}{2}}} v(\xi,\eta)\, d\xi\, d\eta = \sum_{m=-\infty}^{j} \sum_{l=-\infty}^{i} \bar{v}_{lm} \Delta x_l \Delta y_m$$

Reconstruction from cell averages (4).

On a 2D stencil

$$S_{rs}(i,j) = \{(x_{l+1/2}, y_{m+1/2}) : i - r - 1 \leq l \leq i + k - 1 - r,$$

$$j - s - 1 \leq m \leq j + k - 1 - s\}$$

there is a unique polynomial $P(x, y)$ which interpolates V at every point in $S_{rs}(i,j)$.
We take the mixed derivative to get:

$$p(x, y) = \frac{\partial^2 P(x, y)}{\partial x \partial y}$$

Then p approximates $v(x, y)$, which is the mixed derivative of $V(x, y)$, to k-th order:

$$v(x, y) - p(x, y) = O(\Delta^k)$$

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Finite volume formulation.

2D Conservation Law

$$u_t(x,y,t) + f_x(u(x,y,t)) + g_y(u(x,y,t)) = 0$$
$$+ICs + BCs$$

Integrate over a control volume

$$\frac{d\bar{u}_{ij}(t)}{dt} = -\frac{1}{\Delta x_i \Delta y_j}\Big(\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(u(x_{i+\frac{1}{2}},y,t))\,dy - \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(u(x_{i-\frac{1}{2}},y,t))\,dy + \\ + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(u(x,y_{j+\frac{1}{2}},t))\,dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(u(x,y_{j-\frac{1}{2}},t))\,dx\Big)$$

where

$$\bar{u}_{ij}(t) = -\frac{1}{\Delta x_i \Delta y_j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(\xi,\eta,t)\,d\xi d\eta$$

Conservative Scheme.

We approximate the FV formulation by the following conservative scheme:

$$\frac{d\bar{u}_{ij}(t)}{dt} = -\frac{1}{\Delta x_i}(\hat{f}_{i+1/2,j} - \hat{f}_{i-1/2,j}) - \frac{1}{\Delta y_j}(\hat{g}_{i+1/2,j} - \hat{g}_{i,j-1/2})$$

with numerical flux $\hat{f}_{i+1/2,j}$ defined by:

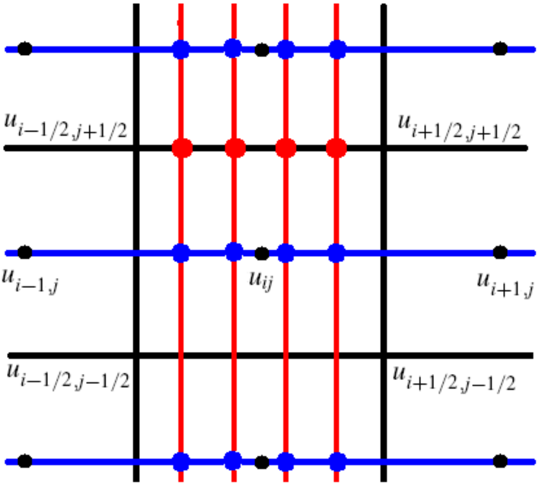
$$\hat{f}_{i+1/2,j} = \sum_{\alpha} \omega_\alpha h(u^-_{i+1/2,y_j+\beta_\alpha \Delta y_j}, u^+_{i+1/2,y_j+\beta_\alpha \Delta y_j})$$

$$\hat{g}_{i,j+1/2} = \sum_{\alpha} \omega_\alpha h(u^-_{x_i+\beta_\alpha \Delta x_i,j+1/2}, u^+_{x_i+\beta_\alpha \Delta x_i,j+1/2})$$

β_α, ω_α - nodes and weights of the Gaussian quadrature for approximating the integrals

$$\frac{1}{\Delta y_j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(u(x_{i+\frac{1}{2}}, y, t))\, dy \text{ and } \frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(u(x, y_{j+\frac{1}{2}}, t))\, dx$$

Gaussian Quadrature Points (Figure).



2D Finite Volume Procedure (Summary).

- Perform the ENO/WENO reconstruction of the values at the Gaussian points $u^{\pm}_{i+1/2, y_j+\beta_\alpha \Delta y_j}$ and $u^{\pm}_{x_i+\beta_\alpha \Delta x_i, i+1/2}$,
- Compute the fluxes $\hat{f}_{i+1/2,j}$ and $\hat{g}_{i,j+1/2}$:

$$\hat{f}_{i+1/2,j} = \sum_\alpha \omega_\alpha h(u^{-}_{i+1/2, y_j+\beta_\alpha \Delta y_j}, u^{+}_{i+1/2, y_j+\beta_\alpha \Delta y_j})$$

$$\hat{g}_{i,j+1/2} = \sum_\alpha \omega_\alpha h(u^{-}_{x_i+\beta_\alpha \Delta x_i, j+1/2}, u^{+}_{x_i+\beta_\alpha \Delta x_i, j+1/2})$$

- Form the scheme:

$$\frac{d\bar{u}_{ij}(t)}{dt} = -\frac{1}{\Delta x_i}(\hat{f}_{i+1/2,j} - \hat{f}_{i-1/2,j}) - \frac{1}{\Delta y_j}(\hat{g}_{i,j+1/2} - \hat{g}_{i,j-1/2})$$

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Reconstruction from point values (1).

Problem formulation

Given the point values of a function $v(x,y)$:

$$v_{ij} \equiv v(x_i, y_j), \quad i = 1, 2, \ldots, N_x, \; j = 1, 2, \ldots, N_y$$

find numerical flux functions:

$$\hat{v}_{i+1/2,j} \equiv \hat{v}(v_{i-r,j}, \ldots, v_{i+k-1-r,j}), \; i = 1, 2, \ldots, N_x$$
$$\hat{v}_{i,j+1/2} \equiv \hat{v}(v_{i,j-s}, \ldots, v_{i,j+k-1-s}), \; j = 1, 2, \ldots, N_y$$

s.t. we get a k-th order approximation of the derivatives:

$$\frac{1}{\Delta x}(\hat{v}_{i+1/2,j} - \hat{v}_{i-1/2,j}) = v_x(x_i, y_j) + O(\Delta x^k), \; i = 1, 2, \ldots, N_x$$
$$\frac{1}{\Delta y}(\hat{v}_{i,j+1/2} - \hat{v}_{i,j-1/2}) = v_y(x_i, y_j) + O(\Delta y^k), \; j = 1, 2, \ldots, N_y$$

Solution: just apply 1D ENO/WENO twice (one direction at a time)

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Finite Difference formulation.

2D Conservation Law

$$u_t(x,y,t) + f_x(u(x,y,t)) + g_y(u(x,y,t)) = 0$$

$$+ICs + BCs$$

We use a conservative approximation to the spatial derivative:

$$\frac{du_{ij}(t)}{dt} = -\frac{1}{\Delta x}(\hat{f}_{i+1/2,j} - \hat{f}_{i-1/2,j}) - \frac{1}{\Delta y}(\hat{g}_{i,j+1/2} - \hat{g}_{i,j-1/2})$$

$u_{ij}(t)$ is the numerical approximation of the point value $u(x_i, y_j, t)$.

2D Finite Difference Procedure.

- Take $v(x) = f(u(x, y_j, t))$ (j fixed)
- Compute $\hat{f}_{i+\frac{1}{2},j}$ using the 1D ENO/WENO procedure for $v(x)$
- Take $v(y) = g(u(x_i, y, t))$ (i fixed)
- Compute $\hat{g}_{i,j+\frac{1}{2}}$ using the 1D ENO/WENO procedure for $v(y)$
- Form the scheme

$$\frac{du_{ij}(t)}{dt} = -\frac{1}{\Delta x}(\hat{f}_{i+1/2,j} - \hat{f}_{i-1/2,j}) - \frac{1}{\Delta y}(\hat{g}_{i,j+1/2} - \hat{g}_{i,j-1/2})$$

Comparison FV ENO/WENO vs. FD ENO/WENO.

	FV ENO/WENO	FD ENO/WENO
Arbitrary meshes	Yes	No
Easy to extend to nD	No	Yes
Operation count (2D)	$4q$	q
Operation count (3D)	$9q$	q

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General Framework.

System of conservation laws

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = 0, \mathbf{U} \in \mathbb{R}^m$$

We consider hyperbolic *m x m* systems, which means the Jacobian matrix $\mathbf{F}'(\mathbf{U})$ has *m* real eigenvalues

$$\lambda_1(\mathbf{U}) \leq \lambda_2(\mathbf{U}) \leq ... \leq \lambda_m(\mathbf{U})$$

and a complete set of independent eigenvectors

$$r_1(\mathbf{U}), r_2(\mathbf{U}), ..., r_m(\mathbf{U})$$

Component-wise FV Procedure

- For each component of the solution vector **U**, apply the scalar ENO/WENO procedure to reconstruct the corresponding component of the solution at cell interfaces, $u^{\pm}_{i+1/2}$ for all i;
- Apply an exact or approximate Riemann solver to compute the numerical flux;
- Form the scheme

$$\frac{d\mathbf{U}}{dt} = -\frac{1}{\Delta x}(\hat{\mathbf{F}}_{i+\frac{1}{2}} - \hat{\mathbf{F}}_{i-\frac{1}{2}})$$

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The Idea of Characteristic Decomposition (1)

System of conservation laws

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = 0, \mathbf{U} \in \mathbb{R}^m$$

For simplicity assume $\mathbf{F}(\mathbf{U}) = A\mathbf{U}$ is linear and A is a constant matrix

$$\mathbf{U}_t + A\mathbf{U}_x = 0$$

In this case the matrices of the spectral decomposition $A = R\Lambda R^{-1}$ are all constant.

From Physical to Characteristic Variables

We define a change of variable

$$\mathbf{V} = R^{-1}\mathbf{U}$$

To get the PDE system for $\mathbf{V}$, we multiply the PDE system by R^{-1} on the left

$$R^{-1}\mathbf{U}_t + R^{-1}A\mathbf{U}_x = 0$$

and insert an identity matrix $I = RR^{-1}$ to get

$$(R^{-1}\mathbf{U}_t) + (R^{-1}AR)(R^{-1}\mathbf{U}_x) = 0$$

where $\Lambda = R^{-1}AR$ is the diagonalized matrix.

Decoupled PDE system

Now, the PDE system becomes decoupled:

$$\mathbf{V}_t + \Lambda \mathbf{V}_x = 0$$

That is, the m equations are independent and each one is a scalar linear advection equation of the form

$$v_t + \lambda_j v_x = 0$$

Thus, we can use the reconstruction techniques for the scalar equations. After we obtain the results, we can "come back" to the physical space $\mathbf{U}$ by computing

$$\mathbf{U} = R\mathbf{V}$$

General Nonlinear System of Conservation Laws

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = 0, \mathbf{U} \in \mathbb{R}^m$$

Write it in the following form:

$$\mathbf{U}_t + \mathbf{F}'(\mathbf{U})\mathbf{U}_x = 0$$

Problem

All the matrices $R(\mathbf{U})$, $R^{-1}(\mathbf{U})$, $\Lambda(\mathbf{U})$ are NOT constant.

Solution

"Freeze" the matrices locally to carry a similar procedure as in the linear flux case.

Characteristic-wise FV Procedure (1)

The following steps must be performed for each space location:

- Compute an average state $\mathbf{U}_{i+1/2}$, using the simple mean

$$\mathbf{U}_{i+1/2} = \frac{1}{2}(\mathbf{U}_i + \mathbf{U}_{i+1})$$

- Compute the right eigenvectors, the left eigenvectors, and the eigenvalues of the Jacobian matrix $\mathbf{F}'(\mathbf{U})$. Denote them by $R = R(\mathbf{U}_{i+1/2})$, $R^{-1} = R^{-1}(\mathbf{U}_{i+1/2})$, $\Lambda = \Lambda(\mathbf{U}_{i+1/2})$;

Characteristic-wise FV Procedure (2)

- Transform all the values $\mathbf{U}$, which are in the potential stencil of the ENO and WENO reconstructions, to the values $\mathbf{V}$:

$$\mathbf{V}_j = R^{-1}\mathbf{U}_j, \quad j \text{ in a neighborhood of } i;$$

- Perform the scalar ENO or WENO reconstruction procedure, for each component of the characteristic variables $\mathbf{V}$, to obtain $\mathbf{V}^{\pm}_{i+1/2}$;
- Compute the numerical flux $\tilde{\mathbf{F}}_{i+1/2}$
- Transform back into physical space $\hat{\mathbf{F}}_{i+1/2} = R\tilde{\mathbf{F}}_{i+1/2}$

Characteristic-wise FD Procedure.

Characteristic-wise Finite Difference schemes can be obtained using a similar procedure.

Two popular schemes of this type are:

- Characteric-wise FD, Roe-type
- Characteric-wise FD, flux splitting

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The Shallow Water Equations.

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x = 0$$

$h(x,t)$ - height of the water
$u(x,t)$ - velocity
In terms of conserved variables:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} u_2 \\ u_2^2 u_1^{-1} + \frac{1}{2}gu_1^2 \end{pmatrix}_x = 0$$

Dam-break problem:

$$u_1(x,0) = h(x,0) = \begin{cases} 100 & \text{if } x \leq 0; \\ 50 & \text{if } x > 0. \end{cases}$$

$$u_2(x,0) = u(x,0)h(x,0) = 0$$

Numerical Solution of SWE using 4th order ENO.

Space discretization
4th order ENO, FD Roe
$\Delta x = 1m$

Time discretization
3rd order RK
$\Delta t = 5ms$

Numerical Solution of SWE using 2nd order ENO.

Space discretization
2nd order ENO, FD Roe
$\Delta x = 1m$

Time discretization
3rd order RK
$\Delta t = 5ms$