ENO and WENO schemes. Further topics and time Integration

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Outline

1. **Short review**
   - ENO/WENO

2. **Further topics**
   - Subcell resolution
   - Other building blocks

3. **Time Integration**
   - TVD Runge-Kutta methods
   - TVD multistep methods
   - Lax-Wendroff procedure

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Solving hyperbolic conservation laws,consider;

- Class of piecewise smooth functions,$v(x)$
- Fixed stencil,$S(i) = \{I_{i-r}, ..., I_{i+s}\}$, approximation may lead to oscillations near the discontinuity
- Adaptive stencil; left shift r

ENO Procedure

- Compute the divided differences, using the cell averages.
- For cell I_i start with the two-point stencil $\tilde{S}(i) = \{x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\}$.
- Add one of the two neighbouring points,using Newton divided difference measure.
- Repeat previous step until a k-th order stencil is obtained.
- Use reconstruction method to compute the approximation to $v(x)$

- In the stencil choosing process, k candidate stencils are considered covering $2k-1$ cells, but only one stencil is used to reconstruct.

To improve the accuracy:

Assume a uniform grid, i.e. $\Delta x_i = \Delta x, i = 1, \dots, N$. Consider the k candidate stencils

$$S_r(i) = \{x_{i-r}, \dots, x_{i-r+k-1}\} \quad r = 0, \dots, k-1.$$

The reconstruction produces k different approximations to the value $v(x_{i+\frac{1}{2}})$:

$$v^{(r)}_{i+\frac{1}{2}} = \sum_{j=0}^{k-1} c_{rj} \bar{v}_{i-r+j} \quad r = 0, \dots, k-1 \tag{1.1}$$

- WENO takes a convex combination of the $v^{(r)}_{i+\frac{1}{2}}$'s.

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- Instead of using the reconstruction polynomial $P_i(x)$ in the shocked cell I_i , find the location of discontinuity inside I_i, say x_s
- Use neighbouring reconstructions $P_{i-1}(x)$ and $P_{i+1}(x)$
- Extend $P_{i-1}(x)$ and $P_{i+1}(x)$ into the cell I_i
- Require that the cell average $\bar{v}_i$ is preserved:

$$\int_{x_{i-\frac{1}{2}}}^{x_s} P_{i-1}(x)dx + \int_{x_s}^{x_{i+\frac{1}{2}}} P_{i+1}(x)dx = \Delta x_i \bar{v}_i$$

Examples

- rational functions
- trigonometric polynomials
- exponential functions
- radial functions

idea

Find suitable smooth indicators similar to Newton divided difference for the polynomial case.

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Consider,
system of ODEs

$$u_t = L(u)$$

resulting from spatial approximation to PDE;

$$u_t = -f(u)_x$$

Construction

Assume: First order Euler forward time stepping

$$u^{n+1} = u^n + \Delta t L(u^n)$$

is stable in a certain norm

$$||u^{n+1}|| \leq ||u^n||$$

under a suitable restriction on Δt : $\Delta t \leq \Delta t_1$

- For higher order in time Runge-Kutta methods,

$$\Delta t \leq c \Delta t_1 \quad c : \text{ CFL coefficient} \tag{3.1}$$

General Runge-Kutta method

$$\begin{aligned} u^{(i)} &= \sum_{k=0}^{i-1} \left(\alpha_{ik} u^k + \Delta t \beta_{ik} L(u^{(k)}) \right), i = 1, \ldots, m \qquad (3.2) \\ u^{(0)} &= u^n, u^{(m)} = u^{n+1} \end{aligned}$$

For $\alpha_{ik} \geq 0, \beta_{ik} \geq 0 \Rightarrow$ convex combination of Euler forward operators

$$\Delta t \text{ replaced by } \frac{\alpha_{ik}}{\beta_{ik}} \Delta t$$

Consistency;

$$\sum_{k=0}^{i-1} \alpha_{ik} = 1.$$

Lemma

The Runge-Kutta method (3.2) is TVD under the CFL coefficient (3.1)

$$c = \min_{i,k} \frac{\alpha_{ik}}{\beta_{ik}}$$

provided; $\alpha_{ik} \geq 0$ and $\beta_{ik} \geq 0$.

Optimal schemes

- Second order TVD Runge-Kutta

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{n+1} &= \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t L(u^{(1)}) \end{aligned}$$

  with CFL $c = 1$.
- Third order TVD R-K

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}) \end{aligned}$$

- Fourth order TVD Runge-Kutta method does not exist with for positive α_{ik} and β_{ik}

Consider

$\alpha_{ik} \geq 0$ and β_{ik} negative

- introduce an adjoint operator $\tilde{L}$
  - it approximates the same spatial derivatives as L
  - TVD for first order Euler, backward in time:

$$u^{n+1} = u^n - \Delta t \tilde{L} u^n$$

TVD Runge-Kutta methods

Lemma

The Runge-Kutta method (3.2) is TVD under the CFL coefficient (3.1)

$$c = \min_{i,k} \frac{\alpha_{ik}}{|\beta_{ik}|}$$

provided; $\alpha_{ik} \geq 0$ and L replaced by $\tilde{L}$

TVD Runge-Kutta methods

Fourth order TVD R-K

$$
\begin{aligned}
u^{(1)} &= u^n + \frac{1}{2}\Delta t L(u^n) \\
u^{(2)} &= \frac{649}{1600}u^{(0)} - \frac{10890423}{25193600}\Delta t\tilde{L}(u^n) + \frac{951}{1600}u^{(1)} + \frac{5000}{7873}\Delta t L u^{(1)} \\
u^{(3)} &= \frac{53989}{2500000}u^n - \frac{102261}{5000000}\Delta t\tilde{L}(u^n) + \frac{4806213}{20000000}u^{(1)} - \frac{5121}{20000}\Delta t\tilde{L}(u^{(1)}) \\
&\quad + \frac{23619}{32000}u^{(2)} + \frac{7873}{10000}\Delta t L(u^{(2)}) \\
u^{n+1} &= \frac{1}{5}u^n + \frac{1}{10}\Delta t L(u^n) + \frac{6127}{30000}u^{(1)} + \frac{1}{6}\Delta t L(u^{(1)}) + \frac{7873}{30000}u^{(2)} \\
&\quad + \frac{1}{3}u^{(3)} + \frac{1}{6}\Delta t L(u^{(3)})
\end{aligned}
$$

Example

- Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

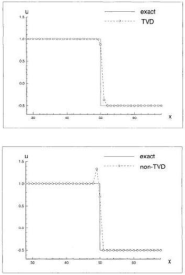
- with Riemann initial data

$$u(x,0) = \begin{cases} 1, & x \le 0 \\ -0.5, & x > 0 \end{cases}$$

- Spatial discretization : 2^{nd} order ENO Scheme
- Monotone flux h: Godunov flux
- Time discretization
    - TVD R-K method :Optimal 2^{nd} order
    - Non-TVD R-K

$$\begin{aligned} u^{(1)} &= u^n - 20\Delta t L(u^n) \\ u^{n+1} &= u^n + \frac{41}{40}u^{(1)} - \frac{1}{40}\Delta t L(u^{(1)}) \end{aligned}$$

TVD Runge-Kutta methods



conclusion

It is much safer to use TVD Runge-Kutta Method for solving hyperbolic problem.

Construction

Similar to Runge-kutta Methods

General From

$$u^{n+1} = \sum_{k=0}^{m-1} \left(\alpha_k u^{n-k} + \Delta t \beta_k L(u^{n-k})\right) \tag{3.3}$$

For $\alpha_k \geq 0, \beta_k \geq 0 \Rightarrow$ convex combination of Euler Forward operators

$$\Delta t \ \ \textit{replaced by} \ \frac{\beta_k}{\alpha_k} \Delta t$$

Consistency;

$$\sum_{k=0}^{m} \alpha_k = 1.$$

TVD multistep methods

Lemma

The multi-step method (3.3) is TVD under the CFL coefficient (3.1)

$$c = \min_k \frac{\alpha_k}{\beta_k}$$

provided; $\alpha_k \geq 0$ and $\beta_k \geq 0$

Examples

- (m=2) three step , second order scheme, TVD with c=0.5

$$u^{n+1} = \frac{3}{4}u^n + \frac{3}{2}\Delta t L(u^n) + \frac{1}{4}u^{n-2} \tag{3.4}$$

- (m=4) five step,third order scheme, TVD with c=0.5

$$u^{n+1} = \frac{25}{32}u^n + \frac{25}{16}\Delta t L(u^n) + \frac{7}{32}u^{n-4} + \frac{5}{16}\Delta t L(u^{n-4})$$

There are no multi-step schemes of order four or higher satisfying positive α_k and β_k

Lemma

The multi-step method (3.3) is TVD under the CFL coefficient (3.1)

$$c = \min_k \frac{\alpha_k}{|\beta_k|}$$

provided; $\alpha_k \geq 0$ and , L is replaced by $\tilde{L}$ for β_k negative.

- one residue evaluation is needed per time step
- storage requirement is much bigger that Runge-Kutta methods

Idea

Taylor series expansion in time:

$$u(x, t + \Delta t) = u(x, t) + u_t(x, t)\Delta t + u_{tt}(x, t)\frac{\Delta t^2}{2} + \ldots \tag{3.5}$$

Illustration

$$u_t = -f(u)_x$$

- second order Taylor series expansion
- Replace time derivatives by the spatial derivatives

Lax-Wendroff procedure

$$\begin{aligned} u_t(x,t) &= -f(u(x,t))_x = -f'(u(x,t))u_x(x,t) \\ u_{tt}(x,t) &= 2f'(u(x,t))f''(u(x,t))(u_x(x,t))^2 + (f'(u(x,t)))^2 u_{xx}(x,t) \end{aligned} \tag{3.6}$$

- Substituting (3.6) into (3.5)
- Discretize the spatial derivatives of $u(x,t)$ by ENO/WENO
- Integrate the pde

  $$u_t(x,t) + f_x(u(x,t)) = 0$$

  over the region $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$ to obtain

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{1}{\Delta x_i}\left(\int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}}, t))\,\mathrm{d}t - \int_{t^n}^{t^{n+1}} f(u(x_{i-\frac{1}{2}}, t))\,\mathrm{d}t\right) \tag{3.7}$$

- Discretize time using suitable Gaussian quadrature

$$\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}}, t))\,\mathrm{d}t \approx \sum_{\alpha} \omega_\alpha f(u(x_{i+\frac{1}{2}}, t^n + \beta_\alpha \Delta t))$$

Lax-Wendroff procedure

- replacing $f(u(x_{i+\frac{1}{2}}, t^n + \beta_\alpha \Delta t))$ by a monotone flux

$$f(u(x_{i+\frac{1}{2}}, t^n + \beta_\alpha \Delta t)) \approx h(u(x^{\pm}_{i+\frac{1}{2}}, t^n + \beta_\alpha \Delta t)$$

- use the Lax-Wendroff procedure to convert

$$u(x^{\pm}_{i+\frac{1}{2}}, t^n + \beta_\alpha \Delta t) \tag{3.8}$$

to $u(x^{\pm}_{i+\frac{1}{2}}, t^n)$ and its spacial derivatives at t^n.

Lax-Wendroff procedure

limitations

- Algebra is very complicated for multi dimensional systems
- difficult to prove stability properties for higher order methods