Polynomial Approximation: The Fourier System

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Outline

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Consider the function,
$u(x) = \sum_{k=-\infty}^{\infty} u_k \phi_k$
expanded in terms of an infinite sequence of orthogonal functions.
Some results:

- periodic functions expanded in Fourier series

We observe that

- decay of the kth coefficient of the expansion faster than the inverse power of k, for infinitely smooth u and all the derivatives are periodic as well.
- Decay not exhibited immediately.

We can imagine that:

- Truncation after a few more steps represents a better approximation
- Spectral accuracy of the Fourier method.
- attainable for non periodic functions provided the expansion functions are chosen properly
- expansion introduces a linear transformation between u and the sequence of its expansion coefficients$(\hat{u}_k)$.

$\hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx, k = 0, \pm 1, \pm 2, ...$ is the Fourier coefficient of u.
It is called the transform of u between physical space and transform (wavenumber) space.

We will try to find:

- Which orthogonal systems are spectral accuracy guaranteed?
- What approximation properties?
- How to use the approximation functions?

Definition:
The Fourier series of a function u is defined as

$$Su = \sum_{k=-\infty}^{\infty} \hat{u}_k \phi_k.$$

ϕ_k are the orthogonal functions.
It represents the formal expansion of u in terms of the Fourier orthogonal system.

Outline

The set of functions

$$\phi_k(x) = e^{ikx}$$

is an orthogonal system over the interval $(0, 2\pi)$.
i.e

$$\int_0^{2\pi} \phi_k(x)\overline{\phi_l(x)}dx = 2\pi\delta_{kl} = \begin{cases} 0 & \text{if } k \neq l \\ 2\pi & \text{if } k = l \end{cases}$$

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The problems encountered are:

- When and in what sense is the series convergent?
- What is the relation between the series and the function u?
- How rapidly does the series converge?

Approximation of Su by the sequence of the trigonometric polynomials $P_N u(x)$:

$$P_N u(x) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k e^{ikx} \quad as\ N \to \infty$$

$P_N u$ is the N-th order truncated Fourier series of u.

Assumptions on the function u:

- Periodic function in $(0, 2\pi)$
- Bounded variation of u on $[0, 2\pi]$
- Uniform convergence of Su
- Pointwise convergence of $P_N u(x)$ to $(u(x^+) + u(x^-))/2$

If u is continuous and periodic, then its Fourier series not necessarily converge at every point $x \in [0, 2\pi]$.
Su is convergent in the mean (L^2 convergent to u) if

$$\int_0^{2\pi} |u(x) - P_N u(x)|^2 dx \to 0 \quad as \quad N \to \infty$$

$L^2(0, 2\pi)$ is a complex Hilbert space.
The inner product and norm are respectively given by

$$(u, v) = \int_0^{2\pi} u(x)\overline{v(x)}dx$$

and

$$||u|| = \left(\int_0^{2\pi} |u(x)|^2 dx\right)^{\frac{1}{2}}$$

Define the space of trigonometric polynomials of degree N/2 as:

$$S_N = span(e^{ikx} | -N/2 \leq k \leq N/2 - 1)$$

By orthogonality,

$$(P_N u, v) = (u, v), \quad \forall v \in S_N.$$

$P_N u$ is the orthogonal projection of u upon the space S_N

Its Fourier series converges to u in mean and by Parsevals identity,

$$\|u\|^2 = 2\pi \sum_{k=-\infty}^{\infty} |\hat{u}_k|^2$$

The numerical series is convergent and conversely, for any complex sequence c_k such that

$$\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$$

there exist a unique solution, $u \in L^2(0, 2\pi)$.
For any function u, we can write

$$u = \sum_{k=-\infty}^{\infty} \hat{u}_k \phi_k$$

By the Riesz theorem, the finite Fourier transform is an isomorphism between $L^2(0, 2\pi)$ and the space ℓ^2 of complex sequences such that

$$\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$$

On the problem of convergence of the series, set:

$$\sum_{|k| \gtrsim N/2} \equiv \sum_{\substack{k < -N/2 \\ k \geq N/2}}$$

By Parsevals identity,

$$\|u - P_N u\| = \left(2\pi \sum_{|k| \gtrsim N/2} |\hat{u}_k|^2 \right)^{1/2}$$

For sufficiently smooth u, then

$$\max_{0\leq x\leq 2\pi} |u(x) - P_N u(x)| \leq \sum_{|k|\gtrsim N/2} |\hat{u}_k|$$

For u, continuously differentiable on the domain and $k \neq 0$

$$2\pi\hat{u}_k = \int_0^{2\pi} u(x)e^{-ikx}dx = -\frac{1}{ik}(u(2\pi^-)-u(0^+))+\frac{1}{ik}\int_0^{2\pi} u'(x)e^{-ikx}dx$$

Hence,

$$\hat{u}_k = 0(k^{-1})$$

and iterating this argument, we have that if u is m-times continuously differentiable, then

$$\hat{u}_k = 0(k^{-m}), \quad k = \pm 1, \pm 2, ...$$

The k-th Fourier coefficient of a function decays faster its negative powers.

Example: The function

$$u(x) = \sin(x/2)$$

is infinitely differentiable in $[0, 2\pi]$ but $u'(0^+) \neq u'(2\pi^-)$ with

$$\hat{u}_k = \frac{2}{\pi}\frac{1}{1-4k^2}$$

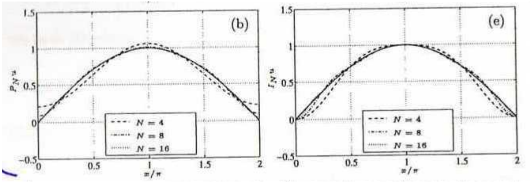

Figure: 1: Trigonometric approximation for $u(x) = sin(x/2)$

Outline

Problems encountered when dealing with the continuous Fourier transformation:

- numerical methods not easy to implement.
- coefficients not known in closed form
- efficient way to recover in physical space, the information calculated in transform space

We use the discrete Fourier transform: For $N > 0$, consider the set of points(nodes or grid points)

$$x_j = \frac{2\pi j}{N}, \quad j = 0, ..., N-1$$

The discrete Fourier coefficients are given by

$$\tilde{u}_k = \frac{1}{N}\sum_{j=0}^{N-1} u(x_j)e^{-ikx_j}, \quad k = -N/2, ..., N/2-1$$

The inversion formula gives

$$u(x_j) = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k e^{ikx_j}, \qquad j = 0, ..., N-1$$

Thus

$$I_N u(x) = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k e^{ikx}$$

is the N/2 -degree trigonometric interpolant of u at the nodes. ie

$$I_N u(x_j) = u(x_j), \qquad j = 0, ..., N-1.$$

It is the discrete Fourier series of u, and the $\tilde{u}_k s$ depend on the values of u at the nodes.
Example in figure 1(e).

The discrete Fourier transform(DFT) is the mapping between $u(x_j)$ and $\tilde{u}_k$.
Another form of the interpolant:

$$I_N u(x) = \sum_{j=0}^{N-1} u(x_j)\psi_j(x)$$

with

$$\psi_j(x) = \frac{1}{N}\sum_{k=-N/2}^{N/2-1} e^{ik(x-x_j)}$$

ψ_j are the trigonometric polynomials in S_N (the characteristic Lagrange trig. polynomials at the nodes) that satisfy

$$\psi_j(x_l) = \delta_{lj}, \qquad l, j = 0, 1, ..., N-1$$

The interpolation operator I_N can be seen as an orthogonal projection upon the space S_N w.r.t the inner product.

The bilinear form

$$(u, v)_N = \frac{2\pi}{N} \sum_{j=0}^{N-1} u(x_j)\overline{v(x_j)}$$

coincides with the inner product if u and v are polynomials of degree N/2.
By orthogonality:

$$(u, v)_N = (u, v), \quad \forall u, v \in S_N$$

and norm

$$||u|| = \sqrt{(u, u)_N} = \sqrt{(u, u)} = ||u||$$

The interpolant satisfies:

$$(I_N u, v)_N = (u, v) \quad \forall v \in S_N$$

The DFC in terms of exact Fourier coefficient of u is given by:

$$\tilde{u}_k = \hat{u}_k + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \hat{u}_{k+Nm}, \qquad k = -N/2, \ldots.., N/2-1$$

This shows aliasing since $\phi_{k+Nm}(x_j) = \phi_k(x_j)$.
The $(k+Nm)$-th wavenumber aliases the k-th wavenumber on the grid.

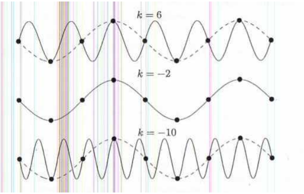

Figure: 2.Three sine waves showing aliasing at k = -2

An equivalent formulation

$$I_N u = P_N u + R_N u,$$

with

$$R_N u = \sum_{k=-N/2}^{N/2-1} \left(\sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \hat{u}_k + Nm \right) \phi_k$$

$R_N u$ is called the aliasing error and it is orthogonal to the truncation error, $u - P_N u$, so that

$$\|u - I_N u\|^2 = \|u - P_N u\|^2 + \|R_N u\|^2$$

This shows that the error due to interpolation is always larger than the error due to truncation.

The influence of the aliasing on the accuracy of the spectral methods is asymptotically of the same order as the truncation error.

Convergence properties of $I_N u$

- uniform convergence on $[0, 2\pi]$
- uniformly bounded on $[0, 2\pi]$ and pointwise convergence to u at every continuity point for u.
- If u is Riemann integrable, then $I_N u$ converges to u in the mean
- for the discrete Fourier coefficients with $\tilde{u}_k = \tilde{u}_k^N$ decays faster than algebraically in k^{-1}, uniformly in N.

Outline

In transform space, If

$$Su = \sum_{k=\infty}^{\infty} \hat{u}_k \phi_k$$

is the Fourier series of u, then

$$Su' = \sum_{k=\infty}^{\infty} ik\hat{u}_k \phi_k$$

is the Fourier series of the derivative of u.

$$(P_N u)' = P_N u'$$

i.e: truncation and differentiation commute.
This is the Fourier projection derivative.

In physical space, the approximate derivative at the grid points are given by

$$(D_N u)_j = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k^{(1)} e^{2ikj\pi/N}, \qquad j = 0, 1, ..., N-1$$

where

$$\tilde{u}_k^{(1)} = ik\tilde{u}_k = \frac{ik}{N} \sum_{l=0}^{N-1} u(x_l) e^{-2ikl\pi/N}, \qquad k = -N/2, ..., N/2-1$$

with

$$D_N u = (I_N u)'$$

In general,

$$D_N u \neq P_N u'$$

The function $D_N u$ is called the Fourier interpolation derivative of u.

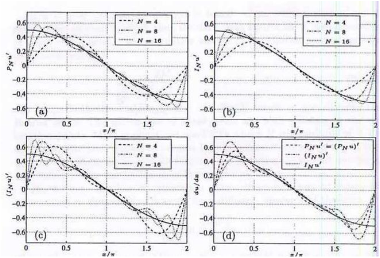

Figure: 3. Fourier differentiation for $u(x) = sin(x/2)$

Interpolation and differentiation do not commute, i.e $(I_N u)' \neq I_N(u')$ unless $u \in S_N$.

The error

$$(I_N u)' - I_N(u')$$

is of the same order as the truncation error for the derivative

$$u' - P_N u'$$

This shows that interpolation differentiation is spectrally accurate.
For $u \in S_N$, $D_N u = u'$,
and
D_N is a skew symmetric operator on S_N:

$$(D_N u, v)_N = -(u, D_N v)_N, \quad \forall u, v \in S_N$$

- Fourier interpolating differentiation represented by a matrix:

The Fourier interpolation derivative matrix is given by

$$(D_N u)_j = \sum_{l=0}^{N-1} (D_N)_{jl} u_l$$

with

$$(D_N)_{jl} = \frac{1}{N} \sum_{k=-N/2}^{N/2-1} ik e^{2ik(j-l)\pi/N}$$

This shows that

$$(D_N)_{jl} = \psi_l'(x_j)$$

The entries of $(D_N)_{jl}$ are the derivatives of the characteristic Lagrange polynomials at the nodes.

Outline

- characteristic oscillatory behaviour of the TFS or DFS in the neighbourhood of a point of discontinuity.
- observed in square waves

Assume, truncation is symmetric w.r.t. N i.e set

$$P_N u = \sum_{|k| \leq N/2} \hat{u}_k \phi_k$$

with

$$P_N u(x) = \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{|k| \leq N/2} e^{-ik(x-y)} \right] u(y) dy$$

Integral representation $P_N u$ is given by

$$P_N u(x) = \frac{1}{2\pi} \int_0^{2\pi} D_N(x-y) u(y) dy$$

Definition: Dirichlet kernel is a collection of functions

$$D_N(\xi) = 1 + 2\sum_{k=1}^{N/2} cosk\xi = \begin{cases} \frac{sin((N+1)\xi/2)}{sin(\xi/2)} & \text{if } \xi \neq 2j\pi \\ & j \in \mathbb{Z} \\ N+1 & \text{if } \xi = 2j\pi \end{cases}$$

Convolution of the Dirichlet kernel with a function $f(x)$ is

$$(D_N * f)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y) D_N(x-y) dy$$

This is the same as the integral representation of $P_N u$.
D_N is an even function that changes sign at the points $\xi_j = 2j\pi(N+1)$ and satisfies $\int_0^{2\pi} D_N(\xi) d\xi = 1$ with $|D_N(\xi)| < \epsilon$ if $N > N(\epsilon, \delta)$ and $\delta \leq \xi \leq 2\pi - \delta, \quad \forall \delta, \epsilon > 0$.

For the square wave, shift the origin to the point of discontinuity, i.e

$$\phi(x) = \begin{cases} 1 & 0 \leq x < \pi \\ 0 & \pi \leq x < 2\pi \end{cases}$$

The TFS is

$$\begin{aligned} P_N\phi(x) &= \frac{1}{2\pi}\int_{x-\pi}^{x} D_N(y)dy \\ &= \frac{1}{2\pi}\left[\int_0^x D_N(y)dy + \int_{-\pi}^{0} D_N(y)dy + \int_{x-\pi}^{-\pi} D_N(y)dy\right] \end{aligned}$$

Hence

$$P_N\phi(x) \simeq \frac{1}{2} + \frac{1}{2\pi}\int_0^x D_N(y)dy. \quad \mathbb{N} \to \infty$$

This formula explains the the Gibbs phenomenon for the square wave.

If$x > 0$, $P_N\phi(x)$ is close to 1.
But $\frac{1}{2\pi}\int_0^x D_N(y)dy$ has alternating maxima and minima at the points where D_N vanishes, this accounts for the oscillation.
For $u = u(x)$, having a jump discontinuity at $x = x_0$, we can write

$$u(x) = u(\tilde{x}) + j(u; x_0)\phi(x - x_0)$$

with

$$P_N u(x) \simeq \frac{1}{2}[u(x_0^+)+u(x_0^-)]+\frac{1}{2\pi}[u(x_0^+)-u(x_0^-)]\int_0^{x-x_0} D_N(y)dy, N \to \infty$$

We see that there is a Gibbs phenomenon at $x = x_0$.

Outline

We focus on the smoothing procedure that attenuate the higher order coefficients.

- multiply each Fourier coefficient $\hat{u}_k$ by a factor σ_k
- Replace $P_N u$ with $S_N u$, the smoothed series.
- the Cesaro sums(take arithmetic means of the truncated series)
- the Lanczos smoothing
- the raised cosine smoothing

The Cesaro sums:

$$S_N u = \frac{1}{N/2+1} \sum_{k=0}^{N/2} P_k u = \sum_{-N/2}^{N/2} \left(1 - \frac{|k|}{N/2+1} \right) \hat{u}_k e^{ikx}$$

$\sigma_k = \frac{sin(2k\pi/N)}{2k\pi/N}, \quad k = -N/2, ..., N/2$ and
$\sigma_k = \frac{1+cos(2k\pi/N)}{2}, \quad k = -N/2, ..., N/2$ are the Lanczos and raised cosine factors respectively

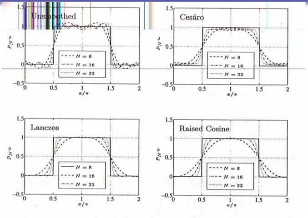

Figure: 4.Several smoothing of the square wave

The smoothed series can be represented in terms of a singular integral as

$$S_N u(x) = \frac{1}{2\pi} \int_0^{2\pi} K_N(x-y)u(y)dy$$

with

$$K_N(\xi) = 1 + 2\sum_{k=1}^{N/2} \sigma_k cos k\xi.$$

The only requirement here is that K_N be an approximate polynomial delta function such that

$$\frac{1}{2\pi}\int_0^{2\pi} K_N(\xi)d\xi = 1$$

Strategy to design a smoothing operator:
Choose the smoothing factors σ_k as

$$\sigma_k = \sigma(2k\pi/N), \quad k = -N/2, ..., N/2$$

where $\sigma = \sigma(\theta)$ called a filtering factor of order p is a real, even function satisfying:

- σ is (p-1) times continuously differentiable in $\mathbb{R}$ for some $p \geq 1$
- $\sigma(\theta) = 0$ if $|\theta| \geq \pi$
- $\sigma(0) = 1$, $\sigma^{(j)}(0) = 0$ for $1 \leq j \leq p - 1$

Some other proposals have being given for the cure of Gibbs phenomenon. The general idea is that whenever the location of singularities are known, the unsmoothed coefficients contain enough information to allow for the construction of an accurate non-oscillatory approximation of the function.