SPECTRAL METHODS: *ORTHOGONAL POLYNOMIALS*

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Preview of Spectral Methods

What are Spectral Methods?

The main components for their formulation are

- Trial functions
- Test functions

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- Galerkin
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- Tau

What choice for the trial function $\omega(x)$?

- Periodic Problem :$\omega(x) \mapsto$ Trigonometric Polynomials
- Non-Periodic Problem :$\omega(x) \mapsto$Orthogonal Polynomials

Orthogonal Polynomials

Sturm -Liouville problems (SLP)

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- A Sturm - Liouville problem is an eigenvalue problem of the form

$$-(\mathrm{p}\mathrm{u}')' + \mathrm{q}\mathrm{u} = \lambda\omega\mathrm{u}$$

in the interval $(-1, 1)$ with boundary condition for u where p, q and ω are given $\mathrm{p} \in C^1(-1, 1), q$ is bounded, ω is the weight function

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- How is spectral accuracy guaranteed?

Properties of Orthogonal Polynomials

- Given $(-1,1)$ and weight $\omega(x) > 0$ on $(-1,1)$ and $\omega \in L^1(-1,1)$. The weighted Sobolev $\mathrm{L}^2{}_\omega(-1,1)$ is defined by

$$\mathrm{L}^2_\omega(-1,1) = \left\{ \mathrm{p} : \int_{-1}^{1} \mathrm{p}^2(x)\omega(x)\mathrm{d}x < +\infty \right\}$$

The inner product of $\mathrm{L}^2_\omega(-1,1)$is given by

$$(\mathrm{p}, g)_\omega = \int_{-1}^{1} \mathrm{p}(x)g(x)\omega(x)\mathrm{d}x$$

and the norm

$$\|\mathrm{p}\|_{\mathrm{L}^2_\omega} = (\mathrm{p},\mathrm{p})_\omega^{1/2}$$

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Properties of Orthogonal Polynomials

- A system of algebraic polynomials $\{\mathrm{p_k}\}_{\mathrm{k}=0,1\ldots}$ with degree k is said to be orthogonal in $\mathrm{L}^2_\omega(-1,1)$ if $(\mathrm{p_k},\mathrm{p_m})_\omega = 0, \quad m \neq \mathrm{k}$ ie

$$\int_{-1}^{1} \mathrm{p_k}(x)\mathrm{p_m}(x)\omega(x)\mathrm{d}x = 0 \text{ whenever } \quad m \neq \mathrm{k}$$

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$$\int_{-1}^{1} \mathrm{p_k}(x)\mathrm{p_m}(x)\omega(x)\mathrm{d}x = 0 \text{ whenever } \quad m \neq \mathrm{k}$$

- A series of a function $\mathrm{u} \in \mathrm{L}^2_\omega(-1,1)$ can be represented in terms of the system $\{\mathrm{p_k}\}$ by

$$\mathrm{Su} = \sum_{\mathrm{k}=0}^{\infty} \hat{u}_\mathrm{k}\mathrm{p_k}$$

where

$$\hat{\mathrm{u}}_\mathrm{k} = \frac{1}{\|\mathrm{p_k}\|^2_\omega}\int_{-1}^{1} \mathrm{u}(x)\mathrm{p_k}(x)\omega(x)\mathrm{d}x$$

is the polynomial transform of u

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$$\hat{\mathrm{u}}_{\mathrm{k}} = \frac{1}{\|\mathrm{p_k}\|^2_\omega}\int_{-1}^{1} \mathrm{u}(x)\mathrm{p_k}(x)\omega(x)\mathrm{d}x$$

is the polynomial transform of u

Existence and Uniqueness of Orthogonal Polynomials

Lemma

If a sequence of polynomials $\{\mathrm{p}_\mathrm{k}\}_{\mathrm{k}=0}^{\infty}$ *is orthogonal then the polynomial* $\mathrm{p}_{\mathrm{N}+1}(x)$ *is orthogonal to any polynomial* q *of degree* N *or less*

Existence and Uniqueness of Orthogonal Polynomials

Theorem

For any positive function $\omega(x) \in \mathrm{L}^1(-1,1)$, $\exists$ *a unique set of Monic orthogonal polynomials* $\{\mathrm{p_k}\}$, *which can be constructed as follows*

$$\mathrm{p}_0 = 1, \mathrm{p}_1 = x - \alpha_1 \textit{with} \;\; \alpha_1 = \int_{-1}^{1} \omega(x) x \mathrm{d}x \Big/ \int_{-1}^{1} \omega(x) \mathrm{d}x$$

and

$$p_{\mathrm{k}+1}(x) = (x - \alpha_{\mathrm{k}+1})\mathrm{p_k}(x) - \beta_{\mathrm{k}+1}\mathrm{p_{k-1}}(x) \;\; \mathrm{k} \geq 1$$

where

$$\alpha_{\mathrm{k}+1} = \int_{-1}^{1} x\omega(x)\mathrm{p_k^2}(x)\mathrm{d}x \Big/ \int_{-1}^{1} \omega(x)\mathrm{p_k^2}(x)\mathrm{d}x$$

and

$$\beta_{\mathrm{k}+1} = \int_{-1}^{1} x\omega(x)\mathrm{p_k}(x)\mathrm{p_{k-1}}(x)\mathrm{d}x \Big/ \int_{-1}^{1} \omega(x)\mathrm{p_{k-1}^2}(x)\mathrm{d}x$$

The Spectral Representation of Function

Orthogonal projections on the space of the polynomials of degree $\leq \mathrm{N}$,

$$\mathrm{P}_{\mathrm{N}}\mathrm{u} = \sum_{\mathrm{k}=1}^{\mathrm{N}} \hat{\mathrm{u}}_{\mathrm{k}}\mathrm{p}_{\mathrm{k}}$$

The completeness of $\{\mathrm{p}_{\mathrm{k}}\}$

$\Longrightarrow \|\mathrm{u} - \mathrm{P}_{\mathrm{N}}\mathrm{u}\| \longrightarrow 0$ as $\mathrm{N} \longrightarrow \infty \ \forall \ \mathrm{u} \in \mathrm{L}^2_\omega(-1,1)$

How to compute the integral

$$\int_{-1}^{1} \mathrm{u}(x)\mathrm{p}_{\mathrm{k}}(x)\omega(x)\mathrm{d}x = (\mathrm{u}, \mathrm{p}_{\mathrm{k}})_{\omega}$$

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By Gauss Integration.

Let $x_0 < x_1 < \dots x_{\mathrm{N}}$ be the roots of $(\mathrm{N}+1) - th$ orthogonal polynomials $\mathrm{p}_{\mathrm{N}+1}(x)$ and let $\omega_0 \dots \omega_{\mathrm{N}}$ be the solution of the linear system

$$\sum_{j=0}^{\mathrm{N}} (x_j)^{\mathrm{k}}\omega_j = \int_{-1}^{1} x^{\mathrm{k}}\omega(x)\mathrm{d}x; \qquad 0 \le \mathrm{k} \le \mathrm{N}.$$

Then $\omega_j > 0$ for $j = 0, 1 \dots \mathrm{N} >$ and

$$\int_{-1}^{1} x^{\mathrm{k}}\mathrm{p}(x)\omega(x)\mathrm{d}x = \sum_{j=0}^{\mathrm{N}} (x_j)^{k}\omega_j \quad \forall \mathrm{p} \in \mathrm{p}_{2N+1}$$

The Way Out Is As Follows

Gauss Radau Integration

Consider the polynomial

$$q(x) = p_{N+1}(x) + a p_N(x)$$

Let $-1 = x_0 < x_1 \cdots < x_N$ be the $(N+1)$ roots of the polynomial and $\omega_0, \ldots \omega_N$ be the solution of the linear system

$$\sum_{j=0}^{N} (x_j)^k \omega_j = \int_{-1}^{1} x^k \omega(x) dx.$$

Then

$$\int_{-1}^{1} x^k p(x) \omega(x) dx = \sum_{j=0}^{N} (x_j)^k \omega_j \quad \forall \quad p \in p_{2N}$$

The Way Out Is As Follows

Gauss Lobatto Integration

Consider the polynomial

$$\mathrm{q}(x) = \mathrm{P}_{\mathrm{N}+1}(x) + a\mathrm{P}_{\mathrm{N}}(x) + b\mathrm{P}_{\mathrm{N}-1}(x)$$

let $-1 = x_0 < x_1 \cdots < x_{\mathrm{N}} = 1$ be the roots of the polynomial and $\omega_0, \ldots \omega_{\mathrm{N}}$ be the solution of the linear system

$$\sum_{j=0}^{\mathrm{N}} (x_j)^{\mathrm{k}} \omega_j = \int_{-1}^{1} x^{\mathrm{k}} \omega(x) \mathrm{d}x$$

Then

$$\int_{-1}^{1} x^{\mathrm{k}} \mathrm{p}(x) \omega(x) \mathrm{d}x = \sum_{j=0}^{\mathrm{N}} (x_j)^{\mathrm{k}} \omega_j \qquad \forall \mathrm{p} \in \mathrm{P}_{2N-1}$$

Gauss- Lobatto

The Gauss-Lobatto points for the Jacobi Polynomials corresponding to the weight $\omega(x) = (1-x)^{\alpha}(1+x)^{\alpha}$ for $N = 8$ and $-1/2 \leq \alpha \leq 1/2$

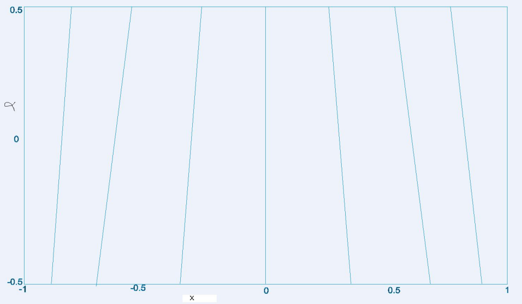

Figure: The Gauss-Lobatto points for $\mathrm{N} = 8$
Jacobi polynomials with the weight function $\omega(x) = (1-x)^{-\alpha}(1+x)^{-\beta}$

The Interpolating Polynomial

The interpolating polynomial associated with $\{x_j\}_{j=0}^{\mathrm{N}}$ $\mathrm{I_N u}$ is defined as a polynomial of degree less than or equal to N, such that

$$\mathrm{I_N u}(x_j) = \mathrm{u}(x_j) \quad j = 0, 1 \ldots \mathrm{N}$$

Hence

$$\mathrm{I_N u} = \sum_{\mathrm{k}=0}^{\mathrm{N}} \tilde{u}_{\mathrm{k}} \mathrm{p_k}$$

The discrete polynomial coefficients of u and its inverse relationship is

$$\tilde{\mathrm{u}}_{\mathrm{k}} = \frac{1}{\gamma_{\mathrm{k}}} \sum_{j=0}^{\mathrm{N}} \mathrm{u}(x_j) \mathrm{p_k}(x_j) \omega_j \qquad \mathrm{k} = 0, \ldots \mathrm{N}$$

The Interpolating Polynomial

The discrete polynomial coefficient $\tilde{\mathrm{u}}_{\mathrm{k}}$ can also be expressed in terms of continuous coefficients $\hat{u}_{\mathrm{k}}$ as

$$\tilde{u}_k = \hat{u}_{\mathrm{k}} + \frac{1}{\gamma_{\mathrm{k}}} \sum_{\ell > \mathrm{N}} (\mathrm{p}_\ell, \mathrm{p}_{\mathrm{k}})_{\mathrm{N}} \hat{\mathrm{u}}_\ell \quad k = 0 \ldots \mathrm{N}$$

$\Longrightarrow \mathrm{I}_{\mathrm{N}} \mathrm{u} = \mathrm{p}_{\mathrm{N}} \mathrm{u} + \mathrm{R}_{\mathrm{N}} \mathrm{u}$ where

$$\mathrm{R}_{\mathrm{N}} \mathrm{u} = \sum_{\mathrm{k}=0}^{\mathrm{N}} \left\{ \frac{1}{\gamma_{\mathrm{k}}} \sum_{\ell > \mathrm{N}} (\mathrm{p}_\ell, \mathrm{p}_{\mathrm{k}})_{\mathrm{N}} \hat{\mathrm{u}}_\ell \right\} \mathrm{p}_{\mathrm{k}}$$

is known as the aliasing error (as a result of the interpolation)

Jacobi Polynomials

- The Jacobi Polynomials is denoted by $\mathrm{J}_n^{\alpha,\beta}(x)$ with $\omega(x) = (1-x)^{\alpha}(1+x)^{\beta}$ for $\alpha, \beta > -1$ on $(-1, 1)$.

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- Also normalized by

$$\mathrm{J}_n^{\alpha,\beta}(1) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\beta+1)}$$

where $\Gamma(x)$ is a gamma function.

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  where $\Gamma(x)$ is a gamma function.
- Satisfies the orthogonality condition

$$\int_{-1}^{1} \mathrm{J}_n^{\alpha,\beta}(x)\mathrm{J}_m^{\alpha,\beta}(x)(1-x)^{\alpha}(1+x)^{\beta})\,\mathrm{d}x = 0 \quad \forall \quad n \neq m$$

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- satisfies the singular Sturm-Liouville Problem

$$(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}}{\mathrm{d}x}\left\{(1-x)^{\alpha+1}(1+x)^{\beta+1}\right\}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{J}_n^{\alpha,\beta}(x) \\ +n(n+1\alpha+\beta)\mathrm{J}_n^{\alpha,\beta}(x) = 0$$

Legendre polynomials

Denoted by $\mathrm{L}_{\mathrm{k}}(x), \mathrm{k} = 0, 1 \ldots$ are eigenfunctions of SLP

$$((1 - x^2)\mathrm{L'}_{\mathrm{k}}(x))' + \mathrm{k}(\mathrm{k}+1)\mathrm{L}_{\mathrm{k}}(x) = 0$$

with

$$\mathrm{p}(\mathrm{x}) = 1 - x^2, \quad \mathrm{q}(\mathrm{x}) = 0, \quad \omega(x) = 1$$

. Normalized by

$$\mathrm{L}_{\mathrm{k}}(x) = \frac{1}{2^{\mathrm{k}}} \sum_{\ell=0}^{[\mathrm{k}/2]} (-1)^{\ell} \binom{\mathrm{k}}{\ell} \binom{2\mathrm{k} - 2\ell}{\mathrm{k}} x^{\mathrm{k}-2\ell}$$

Properties of Legendre polynomials

$$|L_k(x)| \leq 1, \qquad -1 \leq x \leq 1,$$

$$L_k(\pm 1) = (\pm 1)^k,$$

$$|L'_k(x)| \leq \frac{1}{2}(k+1), \qquad -1 \leq x \leq 1,$$

$$\int_{-1}^{1} L^2_k(x)\, dx = \left(k + \frac{1}{2}\right)^{-1}.$$

The expansion of any $u \in L^2_\omega(-1,1)$ in terms of the $L'_k s$ is

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k L_k(x), \quad \hat{u}_k = (k + \frac{1}{2}) \int_{-1}^{1} u(x) L_k(x) dx$$

Discrete Legendre Series

The explicit formulas for the quadrature points and weights are

- Legendre Gauss(LG)
- Legendre Gauss -Radau(LGR)
- Legendre Gauss -Lobatto(LGL)

Discrete Legendre Series

LG, LGR, LGL

- $$\mathrm{x_j}(\mathrm{j} = 0 \ldots N) \text{ zeros of } \mathrm{L_{N+1}}$$
  $$\omega_j = \frac{2}{(1 - \mathrm{x}_j^2)[\mathrm{L'}_{\mathrm{N+1}}(\mathrm{x_j})]^2} \quad \mathrm{j} = 0 \ldots \mathrm{N}$$

Discrete Legendre Series

LG, LGR, LGL

- $$x_j (j = 0 \dots N) \text{ zeros of } L_{N+1}$$

  $$\omega_j = \frac{2}{(1 - x_j^2)[L'_{N+1}(x_j)]^2} \quad j = 0 \dots N$$

- $$x_j (j = 0 \dots N), \text{zeros of } \; L_N + L_{N+1}$$

  $$\omega_0 = \frac{2}{(N+1)^2}, \quad \omega_j = \frac{1}{(N+1)^2} \frac{1 - x_j}{[L_{N+1}(x_j)]^2}, \quad j = 1, \dots, N.$$

Discrete Legendre Series

LG, LGR, LGL

- $$\mathrm{x_j} (\mathrm{j} = 0 \ldots N) \text{ zeros of } \mathrm{L_{N+1}}$$
  $$\omega_j = \frac{2}{(1 - \mathrm{x}_j^2)[\mathrm{L'}_{\mathrm{N+1}}(\mathrm{x_j})]^2} \quad \mathrm{j} = 0 \ldots \mathrm{N}$$
- $$\mathrm{x}_j (j = 0 \ldots \mathrm{N}), \text{zeros of } \mathrm{L_N} + \mathrm{L_{N+1}}$$
  $$\omega_0 = \frac{2}{(\mathrm{N}+1)^2}, \quad \omega_j = \frac{1}{(\mathrm{N}+1)^2} \frac{1 - \mathrm{x_j}}{[\mathrm{L}_{N+1}(\mathrm{x}_j)]^2}, \quad j = 1, \ldots, \mathrm{N}.$$
- $$\mathrm{x_0} = -1, \mathrm{x_N} = 1, \mathrm{x}_j (j = 0 \ldots \mathrm{N} - 1), \quad \text{zeros of } \mathrm{L'}_\mathrm{N}$$
  $$\omega_j = \frac{2}{(\mathrm{N}+1)} \frac{1}{[\mathrm{L}_N(\mathrm{x}_j)]^2} \quad \text{for all } \mathrm{j} = 0 \ldots \mathrm{N}$$

Discrete Legendre Series Continue

The normalization factor is given by

$$\gamma_k = (k + \frac{1}{2})^{-1} \quad \text{for} \quad k < N$$

$$\gamma_N = \begin{cases} (N + \frac{1}{2}) & \text{for Gauss and Gauss-Radau formulas,} \\ 2/N & \quad \text{for the Gauss-Lobatto formula.} \end{cases}$$

Differentiation of Legendre Polynomials

That is if $u = \sum_{k=0}^{\infty} \hat{u}_k L_k$ then u' can be represented as

$$u' = \sum_{k=0}^{\infty} \hat{u}_k^{(1)} L_k$$

where

$$\hat{u}_k^{(1)} = (2k+1) \sum_{\substack{p=k+1 \\ p+k,\text{odd}}}^{\infty} \hat{u_p} k \geq 0$$

The recursion relation is given by

$$u'(x) = \sum_{k=1}^{\infty} \left[\frac{\hat{u}_{k-1}^{(1)}}{2k-1} - \frac{\hat{u}_{k+1}^{(1)}}{2k+3} \right] L'_k(x)$$

Chebyshev Polynomials

- The Chebyshev Polynomial of the first kind is denoted by $\mathrm{T}_k(x), \mathrm{k} = 0, 1 \dots$ are the eigenfunctions of SLP

$$\left(\sqrt{1-\mathrm{x}^2}\,\mathrm{T'}_{\mathrm{k}}(x)\right)' + \frac{\mathrm{k}^2}{\sqrt{1-\mathrm{x}^2}}\mathrm{T}_{\mathrm{k}}(\mathrm{x}) = 0$$

with $\mathrm{p}(\mathrm{x}) = (1-\mathrm{x}^2)^{\frac{1}{2}}$, $\mathrm{q}(\mathrm{x}) = 0$ and $\omega(\mathrm{x}) = (1-\mathrm{x}^2)^{\frac{-1}{2}}$

- The chebyshev polynomial can be expressed in a power series as

$$\mathrm{T}_k(\mathrm{x}) = \frac{\mathrm{k}}{2}\sum_{\ell=0}^{[k/2]}(-1)^{\mathrm{k}}\frac{(\mathrm{k}-\ell-1)}{\ell!(\mathrm{k}-2\ell)!})2\mathrm{x}^{\mathrm{k}-2\ell}$$

Chebyshev Polynomials

- The trigonometric relation
  $\cos(k+1)\theta + \cos(k-1)\theta = 2\cos\theta\cos k\theta$
  gives the recursion relation

$$T_{k+1}(x) = 2xT_k - T_{k-1}(x)$$

with $T_0(x) \equiv 1$ and $T_1(x) \equiv x$

Properties of Chebyshev Polynomials

$$|\mathrm{T}_k(x)| \leq 1, \quad -1 \leq \mathrm{x} \leq 1,$$

$$\mathrm{T}_k(\pm 1) = (\pm 1)^k,$$

$$|\mathrm{T}'_k(x)| \leq (\mathrm{k}^2), \quad -1 \leq \mathrm{x} \leq 1,$$

$$\mathrm{T}'_k(\pm) = (\pm)^{\mathrm{k}+1}\mathrm{k}^2,$$

$$\int_{-1}^{1} \mathrm{T}^2{}_k(x)\frac{\mathrm{d}x}{\sqrt{1-\mathrm{x}}} = \mathrm{c_k}\frac{\pi}{2},$$

where

$$\mathrm{c_k} = \begin{cases} 2, & \mathrm{k} = 0 \\ 1, & \mathrm{k} \geq 1. \end{cases}$$

The Chebyshev expansion of a function $\mathrm{u} \in \mathrm{L}^2_w(-1,1)$ is

$$\mathrm{u(x)} = \sum_{\mathrm{k}=0}^{\infty} \hat{\mathrm{u}}_\mathrm{k}\mathrm{L_k(x)}, \quad \hat{\mathrm{u}}_\mathrm{k} = \mathrm{c_k}\frac{\pi}{2}\int_{-1}^{1} \mathrm{u(x)T_k(x)d}x$$

Discrete Chebyshev Series

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Discrete Chebyshev Series

Chebyshev Gauss(CG), Chebyshev Gauss -Radau(CGR), Chebyshev Gauss -Lobatto(CGL)

- $$x_j = \cos\frac{(2j+1)\pi}{2N+2}, \quad \omega_j = \frac{\pi}{N+1}, \quad j = 0, \ldots, N$$

Discrete Chebyshev Series

Chebyshev Gauss(CG), Chebyshev Gauss -Radau(CGR), Chebyshev Gauss -Lobatto(CGL)

- $$x_j = \cos\frac{(2j+1)\pi}{2N+2}, \quad \omega_j = \frac{\pi}{N+1}, \quad j = 0, \ldots, N$$

- $$x_j = \cos\frac{2\pi j}{2N+1}, \quad \omega_j = \begin{cases} \frac{\pi}{2N+1}, & j = 0, \\ \frac{\pi}{2N+2}, & j = 0, \ldots, N \end{cases}$$

Discrete Chebyshev Series

Chebyshev Gauss(CG), Chebyshev Gauss -Radau(CGR), Chebyshev Gauss -Lobatto(CGL)

- $$\mathrm{x_j} = \cos\frac{(2j+1)\pi}{2\mathrm{N}+2}, \quad \omega_j = \frac{\pi}{\mathrm{N}+1}, \quad \mathrm{j} = 0, \dots, \mathrm{N}$$
- $$\mathrm{x_j} = \cos\frac{2\pi j}{2\mathrm{N}+1}, \quad \omega_j = \begin{cases} \frac{\pi}{2\mathrm{N}+1}, & \mathrm{j} = 0, \\ \frac{\pi}{2\mathrm{N}+2}, & \mathrm{j} = 0, \dots, \mathrm{N} \end{cases}$$
- $$\mathrm{x_j} = \cos\frac{j\pi}{2\mathrm{N}}, \quad \omega_j = \begin{cases} \frac{\pi}{2\mathrm{N}}, & \mathrm{j} = 0, \mathrm{N}, \\ \frac{\pi}{\mathrm{N}}, & \mathrm{j} = 1, \dots \mathrm{N} - 1 \end{cases}$$

The Chebyshev transform space

This is given by

$$\mathrm{C_k} = \frac{2}{\mathrm{N}\overline{c}_{\mathrm{j}}\,\overline{c}_{\mathrm{k}}} \cos \frac{\pi \mathrm{jk}}{\mathrm{N}}$$

where

$$\overline{\mathrm{c}}_{\mathrm{k}} = \begin{cases} 2, & \mathrm{j} = 0, \mathrm{N}, \\ 1, & \mathrm{j} = 1, \ldots \mathrm{N} - 1 \end{cases}$$

The inverse transform is represented by

$$(\mathrm{C}^{-1})_{\mathrm{jk}} = \cos \frac{\pi \mathrm{jk}}{\mathrm{N}}$$

Both transforms can be evaluated by the Fast Fourier Transform

The normalization factors γ_k is given by

$$\gamma_k = \frac{\pi}{2} c_k \quad \text{for} \ \ k < N$$

$$\gamma_N = \begin{cases} \frac{\pi}{2} & \text{for Gauss and Gauss-Radau formulas,} \\ \pi & \text{for the Gauss-Lobatto formula} \end{cases}$$

The aliasing error for the Chebyshev Gauss-Lobatto points is given by

$$\tilde{u}_k = \hat{u}_k + \sum_{\substack{j=2mN\pm k \\ j>N}} \hat{u}_j$$

Differentiation of Chebyshev polynomials

The derivative of a function u expanded in Chebyshev polynomial is given by

$$\mathrm{u}' = \sum_{\mathrm{k}=0}^{\infty} \hat{\mathrm{u}}_{\mathrm{k}}^{(1)} \mathrm{T}_{\mathrm{k}}$$

where

$$\hat{\mathrm{u}}_{\mathrm{k}}^{(1)} = \frac{2}{\mathrm{c}_{\mathrm{k}}} \sum_{\substack{\mathrm{p}=\mathrm{k}+1 \\ \mathrm{p}+\mathrm{k} \quad \text{odd}}}^{\infty} \mathrm{p}\hat{\mathrm{u}_{\mathrm{p}}} \quad \mathrm{k} \geq 0$$

The above expression is a consequence of the relation

$$2\mathrm{T}_{\mathrm{k}}(x) = \frac{1}{\mathrm{k}+1} \mathrm{T}'_{\mathrm{k}+1}(x) - \frac{1}{\mathrm{k}-1} \mathrm{T}'_{\mathrm{k}-1}(x)$$

and finally we obtain

$$2\mathrm{k}\hat{\mathrm{u}}_{\mathrm{k}} = \mathrm{c}_{\mathrm{k}-1} \hat{\mathrm{u}}_{\mathrm{k}-1}^{(1)} - \hat{\mathrm{u}}_{\mathrm{k}+1}^{(1)}$$

Differentiation of Chebyshev Polynomials Continue

The recursion relation is given by

$$c_k \hat{u}^{(1)} = \hat{u}^{(1)}_{k+2} + 2(k+1)\hat{u}^{(1)}_{k+1}, \qquad 0 \le k \le N-1$$

The generalization relation is given by

$$c_k \hat{u}^{(q)} = \hat{u}^{(q)}_{k+2} + 2(k+1)\hat{u}^{(q-1)}_{k-1}, \qquad k \ge 0$$

The coefficients of the second derivative are

$$\hat{u}^{(2)}_k = \frac{1}{c_k} \sum_{\substack{p=k+2 \\ p+k \text{ even}}}^{\infty} p(p^2-k^2)\hat{u}_p, \quad k \ge 0$$

A simple Differential equation with boundary conditions

Let consider the $1-\mathrm{D}$ second-order linear PDE

$$\frac{\mathrm{d}^2\mathrm{u}}{\mathrm{dx}^2} - 4\frac{\mathrm{du}}{\mathrm{dx}} + 4\mathrm{u} = \mathrm{e}^{\mathrm{x}} + \mathrm{C}, \quad \mathrm{x} \in [-1, 1]$$

with the Dirichlet boundary conditions

$$\mathrm{u}(-1) = 0 \text{ and } \mathrm{u}(1) = 0$$

and where C is a constant: $\mathrm{C} = -4\mathrm{e}/(1+\mathrm{e}^2)$. The exact solution of the system is

$$\mathrm{u}(x) = \mathrm{e}^{\mathrm{x}} - \frac{\sinh 1}{\sinh 2}\mathrm{e}^{2\mathrm{x}} + \frac{\mathrm{C}}{4}$$

Solving by Chebyshev spectral mehthod

Look for a numerical solution by the first Chebyshev polynomials: $\mathrm{T}_0(x), \mathrm{T}_1(x), \mathrm{T}_2(x), \mathrm{T}_3(x)$ and $\mathrm{T}_4(x)$, $N = 4$.
Expand the source $\mathrm{u}(x) = \mathrm{e}^{\mathrm{x}} + \mathrm{C}$ onto the Chebyshev Polynomials

$$\mathrm{p}_4 \mathrm{u}(x) = \sum_{n=0}^{4} \tilde{u}_n \mathrm{T}_{\mathrm{n}}(x)$$

and

$$\mathrm{I}_4 \mathrm{u}(x) = \sum_{n=0}^{4} \hat{\mathrm{u}}_n \mathrm{T}_{\mathrm{n}}(x)$$

with

$$\tilde{u}_n = \frac{2}{\pi(1 + \delta_{0\mathrm{n}})} \int_{-1}^{1} \mathrm{T}_{\mathrm{n}}(x)\mathrm{u}(x) \frac{\mathrm{dx}}{\sqrt{1 - x^2}}$$

and

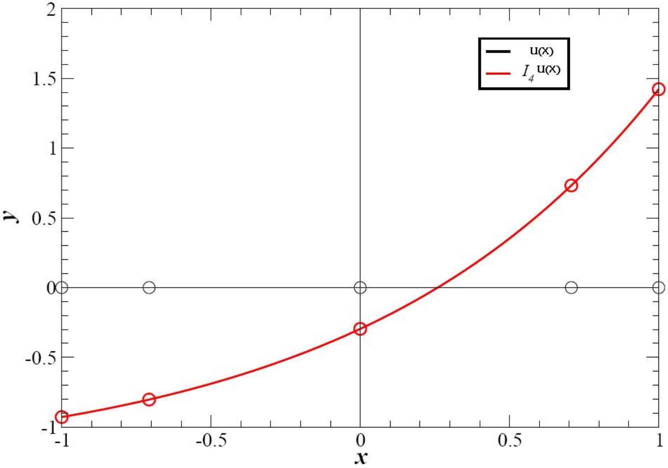
$$\hat{\mathrm{u}}_n = \frac{2}{\pi(1 + \delta_{0\mathrm{n}})} \sum_{n=0}^{4} \mathrm{w}_i \mathrm{T}_{\mathrm{n}}(x_i)\mathrm{u}(x_i)$$

Solving by Chebyshev spectral mehthod

where the $\mathrm{x_i}$'s being the 5 Gauss -Lobatto quadrature points for the weight $\mathrm{w} = (1 - x^2)^{-1/2}$:
$\mathrm{x}_i = \{-\cos(i\pi/4), 0 \leq i \leq 4\} = \{-1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\}$
The continuous coefficient is obtained as follows

$\hat{\mathrm{u}}_N$	$\tilde{\mathrm{u}}_N$	$\hat{\mathrm{u}}_N$-$\tilde{\mathrm{u}}_N$
-0.03004	-0.300402	2.010^{-7}
1.130	1.1299968	3.210^{-6}
0.2715	0.271455	4.510^{-5}
0.04488	0.04434	5.410^{-4}
0.005474	0.005473999999	1.010^{-12}

The source and its Chebyshev interpolant



Interpolation error and the aliasing error

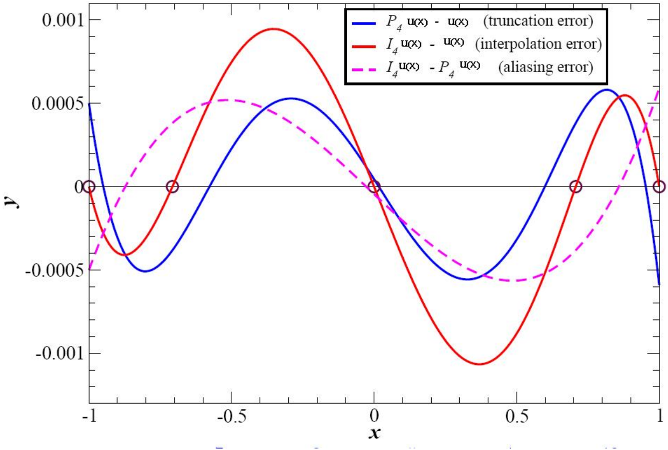


Figure: $\mathrm{N} = 4$(5 Chebyshev polynomials)

——Thank you ————