Spectral Methods for Burger´s Equation

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Outline

Burger´s Equation

Strong Form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \; in\,\Omega, \; \forall t > 0$$

Conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0, \; in\,\Omega, \; \forall t > 0$$

$$F(u) = \frac{1}{2}u^2 - \nu\frac{\partial u}{\partial x}$$

Where: $\nu > 0$ and initial condition $u(x,0) = u_0(x)$.

Solving the equation

- Use Hopf-Cole transformation

$$u = -2\nu \frac{\phi_x}{\phi}$$

- Get a heat equation

$$\frac{\partial \phi}{\partial t} - \nu \frac{\partial^2 \phi}{\partial x^2} = 0$$

- Find solution for periodic and non periodic problems

Analytic Solution

- We get that:

$$u(x,t) = c + u_b(x - ct, t + t_0)$$

where, $u_b = -2\nu \frac{\phi_x}{\phi}$ and

	Nonperiodic	Periodic
$\phi(x,t) =$	$\frac{x}{t}\frac{\sqrt{a/t}e^{-x^2/(4\nu t)}}{1+\sqrt{a/t}e^{-x^2/(4\nu t)}}$	$\frac{1}{\sqrt{4\pi\nu t}}\sum_{n=-\infty}^{\infty} e^{-(x-2\pi n)^2/(4\nu t)}$

Formulations

- Multiply strong form by “test function” v
- For each time, integrate over the space

$$\int_a^b \frac{\partial u}{\partial t} v dx + \int_a^b u \frac{\partial u}{\partial x} v dx - \int_a^b \nu \frac{\partial^2 u}{\partial x^2} v dx = 0, \ v \in X$$

- Using integration by parts

$$\int_a^b \frac{\partial u}{\partial t} v dx - \int_a^b u^2 \frac{\partial v}{\partial x} dx + \int_a^b \nu \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx$$

$$+(\frac{1}{2} u^2 v - \nu \frac{\partial u}{\partial x} v)|_{x=b} - (\frac{1}{2} u^2 v - \nu \frac{\partial u}{\partial x} v)|_{x=a} = 0, \ v \in V$$

Strong Form

$$u_t + G(u) + Lu = 0 \; in\Omega, \; \forall t > 0$$

Integral Form

$$(u_t + G(u) + Lu, v) = 0 \; \forall v \in X$$

Weak Form

$$(u_t, v) - (F(u), v_x) + [F(u)v|_a^b = 0, \; \forall v \in V$$

where, $G(u) = u(\partial u/\partial x)$, $L = -\nu(\partial^2/\partial x^2)$ and F is the flux.

Fourier Galerkin

- **Goal**: find a periodic solution on $(0, 2\pi)$
- Trial space S_N trigonometric polynomials ($deg \leq N/2$)
- Approximate u with u^N given by

$$u^N(x,t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t) e^{ikx}$$

- **New Goal**: determine $\hat{u}_k(t)$
- Using the **integral form** with same tests functions

$$\int_0^{2\pi} \left(\frac{\partial u^N}{\partial t} + u^N \frac{\partial u^N}{\partial x} - \nu \frac{\partial^2 u^N}{\partial x^2}\right) e^{-ikx} dx = 0, k = -N/2, ..., N/2-1$$

- Due to orthogonality of test (trial) functions we get

ODE´s system for $\hat{u}_k$

$$\frac{d\hat{u}_k}{dt} + \left(u^N \frac{\partial u^N}{\partial x}\right)_k^{\wedge} + k^2 \nu \hat{u}_k = 0, \; k = -N/2, ..., N/2 - 1$$

$$\hat{u}_k(0) = \frac{1}{2\pi} \int_0^{2\pi} u(x,0) e^{-ikx} dx$$

- where:

$$\left(u^N \frac{\partial u^N}{\partial x}\right)_k^{\wedge} = \frac{1}{2\pi} \int_0^{2\pi} u^N \frac{\partial u^N}{\partial x} e^{-ikx} dx$$

Fourier Collocation

- **Goal**: find a periodic solution on $(0, 2\pi)$
- Approximate u by its “exact” values at $x_j = 2\pi j/N, j = 0, .., N-1$
- Use the **strong** form at the collocation points

$$\frac{\partial u^N}{\partial t} + u^N \frac{\partial u^N}{\partial x} - \nu \frac{\partial^2 u^N}{\partial x^2}\Big|_{x=x_j} = 0, \ \ u^N(x_j, 0) = u_0(x_j)$$

Strong formulation

$$\frac{d\mathbf{u}}{dt} + \mathbf{u} \otimes D_N \mathbf{u} - \nu D_N^2 \mathbf{u} = 0, \mathbf{u}(t) = (u^N(x_0, t), ..., u^N(x_{N-1}, t))^T$$

- D_N is the Fourier interpolation differentiation matrix and $\otimes$ component-wise product

- We could also use the **conservation** form and get

Conservation Form

$$\frac{d\mathbf{u}}{dt}+\frac{1}{2}D_N(\mathbf{u}\otimes\mathbf{u})-\nu D_N^2\mathbf{u}=0,\mathbf{u}(t)=(u^N(x_0,t),...,u^N(x_{N-1},t))^T$$

- again D_N is the Fourier interpolation differentiation matrix and $\otimes$ component-wise product
- this is **not** equivalent to the one in strong form
- in the Galerkin case the strong and conservation form are equivalent

Chebyshev Tau

- **Goal**: find solution on $(-1,1)$ s.t:

$$u(-1,t) = u_L(t),\ u(1,t) = u_R(t)$$

- Trial space P_N algebraic polynomials ($deg \leq N$)
- Approximate by Chebyshev series

$$u^N(x,t) = \sum_{k=0}^{N} \hat{u}_k(t) T_k(x)$$

- Using the integral form with test functions P_{N-2}

$$\int_{-1}^{1} \left(\frac{\partial u^N}{\partial t} + u^N \frac{\partial u^N}{\partial x} - \nu \frac{\partial^2 u^N}{\partial x^2}\right)(x) T_k(x)(1-x^2)^{1/2} dx = 0,\ k = 0, \dots, N$$

- To find $\hat{u}_k$ we get the following:

ODE´s system

$$\frac{d\hat{u}_k}{dt} + \left(u^N \frac{\partial u^N}{\partial x}\right)^{\wedge}_k - \nu \hat{u}_k^2 = 0, \; k = 0, ..., N-2$$

$$\hat{u}_k(0) = \frac{2}{\pi c_k} \int_{-1}^{1} u_0(x) T_k(x)(1-x^2)^{-1/2} dx$$

- $\hat{u}_k^{(2)} = \frac{1}{c_k} \sum_{p=k+2, p+k even}^{\infty} p(p^2-k^2)\hat{u}_p$ is Chebyshev 2nd-diff,

$$\left(u^N \frac{\partial u^N}{\partial x}\right)^{\wedge}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u^N \frac{\partial u^N}{\partial x}(x) T_k(x)(1-x^2)^{-1/2} dx$$

- Dirichlet BC $\sum_{k=0}^{N} \hat{u}_k = u_R$, $\sum_{k=0}^{N} (-1)^k \hat{u}_k = u_R$
- Neumann BC $\sum_{k=0}^{N} k^2 \hat{u}_k = 0$, $\sum_{k=0}^{N} (-1)^k k^2 \hat{u}_k = 0$

Chebyshev Collocation

- **Goal**: find solution on $(-1, 1)$ s.t. some BC´s
- Approximate u by its "exact" values at $x_j = cos\pi j/N, j = 0, .., N-1$
- Use the **strong** form at the collocation points

$$\frac{\partial u^N}{\partial t} + u^N \frac{\partial u^N}{\partial x} - \nu \frac{\partial^2 u^N}{\partial x^2}|_{x=x_j} = 0,\ \ u^N(x_j, 0) = u_0(x_j)$$

Vector form

$$Z_N(\frac{d\mathbf{u}}{dt} + \mathbf{u} \otimes D_N\mathbf{u} - \nu D_N^2\mathbf{u}) = 0, \mathbf{u}(t) = (u(x_0, t), ..., u(x_N, t))^T$$

- D_N is the Chebyshev interpolation differentiation matrix
- Z_N matrix that makes first and last points zero

Legendre Galerkin-NI

- **Goal**: find solution on $(-1,1)$ s.t. $F(u) = 0$ at $x = \pm 1$
- Use the **weak** form

$$\int_{-1}^{1} \frac{\partial u}{\partial t} v dx - \frac{1}{2} \int_{-1}^{1} u^2 \frac{\partial v}{\partial x} dx + \nu \int_{-1}^{1} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx = 0$$

- Approximate the solution taking P_N as trial and test function spaces$u^N(x,t) = \sum_{l=0}^{N} u_l^N(t)\psi_l(x)$
- Using Gauss Lobatto quadrature formula ($w(x) = 1$)

$$(\frac{\partial u^N}{\partial t}, v)_N - \frac{1}{2}((u^N)^2, \frac{\partial v}{\partial x}) + \nu(\frac{\partial u^N}{\partial x}, \frac{\partial v}{\partial x})_N = 0$$

- Use discrete delta functions as test functions (characteristic Lagrange Polynomials)
- Get ODE system and solve

Fourier Collocation N=16

Figure: $\nu = 0.2$, $c = 4$, $t_0 = 1$, from $t \in [0, \pi/8]$

Fourier Collocation N=32

Figure: $\nu = 0.2$, $c = 4$, $t_0 = 1$, from $t \in [0, \pi/8]$

Fourier Collocation N=64

Figure: $\nu = 0.2$, $c = 4$, $t_0 = 1$, from $t \in [0, \pi/8]$

Long Time N=16

Figure: $\nu = 0.2$, $c = 4$, $t_0 = 1$, from $t \in [0, 6]$

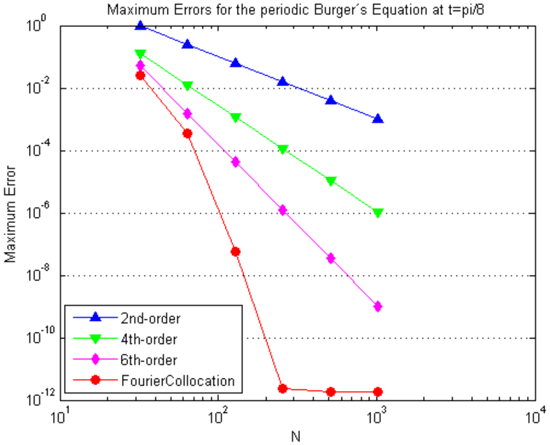
Long Time N=32

Figure: $\nu = 0.2$, $c = 4$, $t_0 = 1$, from $t \in [0, 6]$

Long Time N=64

Figure: $\nu = 0.2$, $c = 4$, $t_0 = 1$, from $t \in [0, 6]$

Error Comparison



2nd, 4th and 6th order data from:"C.Canuto, et al, *Spectral Methods: Fundamentals in Single Domains*"

Thanks for your attention!