One-phase Stefan problems

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Outline

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1. MBP
   - The Stefan Problem
2. Strategies for solving MBP
   - Tentative list of methods
   - Self-similar solutions
3. Global heat balance
   - Dirichlet problem
   - Neumann problem
4. Classical solution
   - Maximum principle
   - Existence theorem

MBP: The Stefan Problem

describes the melting of ice and the freezing of water.

Definition

$Q_T = \Omega \times (0, T) = \{(x, t) \in \mathbb{R}^2 | 0 < x < s(t), 0 < t < T\}$,
find (s,u) such that
$s \in C[0, T] \cap C^1(0, T), \quad u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$,
$u_x(x, t)$ - continuous,

$$u_{xx} - u_t = 0, \text{ in } Q_T,$$

$s(0) = b > 0$,
$u(x, 0) = h(x), 0 < x < s(x)$,
$u(0, t) = f(t), 0 < t < T, \quad | \quad u_x(0, t) = g(t), 0 < t < T$,
$u(s(t), t) = 0, 0 < t < T$,
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Sign restrictions: $h \geq 0$, $f \geq 0$ or $g \leq 0$.

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Solving MBP strategies

- Integral equation methods;
- Mapping methods;
- Fixed-domain methods;
- Alternative approaches (for example, self-similar solutions, etc.).

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Self-similar solutions

Heating problem within a fixed-domain

1. $f(\xi), \gamma(t)$, such that $u(x,t) = f(\gamma(t)x)$ satisfies $u_{xx} = u_t$.
2. $\gamma^2 f'' = \dot{\gamma} x f' \Rightarrow \frac{f''}{\xi f'} = \frac{\dot{\gamma}}{\gamma^3} = \lambda^2$, with $\lambda \in \mathbb{R}$.
3. $\frac{\dot{\gamma}}{\gamma^3} = \lambda^2 \Rightarrow \gamma(t) = \lambda^{-1}[2(t-t_0)]^{-\frac{1}{2}}, t > t_0$.
4. $\frac{f''}{\xi f'} = \lambda^2 \Rightarrow f(\xi) = A(\mathrm{erf}(\frac{\lambda\xi}{\sqrt{2}}) - \mathrm{erf}(\frac{\lambda\xi_0}{\sqrt{2}}))$, where $\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy$.
5. We can define the desired class of self-similar solutions:

$$u(x,t) = A(\mathrm{erf}(\frac{x}{2\sqrt{(t-t_0)}}) - \mathrm{erf}(z_0)),$$

depending on three parameters A, t_0, z_0.

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The Stefan problem

1. We set $t_0 = 0$.
2. $u(s(t),t) = A(\mathrm{erf}(\frac{s(t)}{2\sqrt{t}}) - \mathrm{erf}(z_0)) = 0 \Rightarrow s(t) = 2\alpha\sqrt{t}$.
3. $\frac{\partial u}{\partial x} = A\frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4t}} = -\dot{s}(t) = -\frac{\alpha}{\sqrt{t}} \Rightarrow A = -\sqrt{\pi}\alpha e^{\alpha^2}$.
4. One-parameter family of self-similar solutions:

$$u(x,t) = \sqrt{\pi}\alpha e^{\alpha^2}(\mathrm{erf}(\alpha) - \mathrm{erf}(\frac{x}{2\sqrt{t}})), \quad s(t) = 2\alpha\sqrt{t}.$$

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   - Dirichlet problem
   - Neumann problem
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   - Maximum principle
   - Existence theorem

Global heat balance: Dirichlet problem

Global heat balance

$$\int_0^{s(t)} xu(x,t)dx = \int_0^b xh(x)dx + \int_0^t f(\tau)d\tau - \frac{1}{2}(s^2(t) - b^2).$$

- $u_{xx} - u_t = 0 \Rightarrow$

$$\int_{Q_t} x(u_{xx} - u_\tau)dxd\tau = 0.$$

- $xu_{xx} = (xu_x - u)_x \Rightarrow$

$$\int_{Q_t} [(xu_x - u)_x - (xu)_\tau]dxd\tau = 0.$$

- Green's Theorem $\Rightarrow$

$$\oint_{\partial Q_t} [(xu_x - u)d\tau + xudx] = 0.$$

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$$\int_{(0,t)}^{(s(t),t)} [(xu_x - u)\underbrace{d\tau}_{=0} + xudx] = \int_0^{s(t)} xu(x,t)dx. \quad (2)$$

$$\int_{(s(t),t)}^{(b,0)} [(\underbrace{x}_{=s(\tau)} \underbrace{u_x}_{=-\dot{s}(\tau)} - \underbrace{u}_{=0})d\tau + x \underbrace{u}_{=0} dx] = \int_0^t [s(\tau)\dot{s}(\tau)]d\tau = \frac{1}{2}(s^2(t) - b^2). \quad (3)$$

$$\int_{(b,0)}^{(0,0)} [(xu_x - u)\underbrace{d\tau}_{=0} + x \underbrace{u}_{=h(x)} dx] = -\int_0^b xh(x)dx. \quad (4)$$

Global heat balance: Neumann problem

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$$\int_0^{s(t)} u(x,t)dx = \int_0^b h(x)dx - \int_0^t g(\tau)d\tau - (s(t) - b).$$

- $u_{xx} - u_t = 0 \Rightarrow$

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$$\int_{(0,0)}^{(0,t)} [\underbrace{u_x}_{g(\tau)} d\tau + u \underbrace{dx}_{=0}] = \int_0^t g(\tau) d\tau. \quad (5)$$

$$\int_{(0,t)}^{(s(t),t)} [u_x \underbrace{d\tau}_{=0} + u dx] = \int_0^{s(t)} u(x,t) dx. \quad (6)$$

$$\int_{(s(t),t)}^{(b,0)} [\underbrace{u_x}_{=-\dot{s}(\tau)} d\tau + \underbrace{u}_{=0} dx] = \int_0^t \dot{s}(\tau) d\tau = s(t) - b. \quad (7)$$

$$\int_{(b,0)}^{(0,0)} [u_x \underbrace{d\tau}_{=0} + \underbrace{u}_{=h(x)} dx] = -\int_0^b h(x) dx. \quad (8)$$

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Maximum principle

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- **Strong version**: if u has it's maximum [minimum] on $\{\overline{Q}_T \backslash \partial_P Q_T\}$, then u is constant in $\overline{Q}_T$.

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Positivity of the temperature and the interface velocity

Proposition

For any solution of **our** one-phase Stefan problem

1. $u \geq 0$;
2. $\dot{s} \geq 0$.

1. - BC's, sign restrictions $\Rightarrow \min_{\partial_P Q_T} u = 0$.
   - Maximum principle (the strong version) $\Rightarrow$

$$\min_{\overline{Q_T}} u = \min_{\partial_P Q_T} u = 0 \Rightarrow u \geq 0 \text{ in } \overline{Q_T}.$$

2. $u \geq 0 \in \overline{Q_T}$ and $x < s(t) \Rightarrow$

$$s'(t) = -\lim_{x \to s(t)^-} \frac{u(x,t) - u(s(t),t)}{x - s(t)} = -\lim_{x \to s(t)^-} \frac{u(x,t)}{x - s(t)} \geq 0, \forall t.$$

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Existence theorem

We refer to the case

$$s(0) = b > 0 \tag{9}$$

and assume that $f(x), h(x)$ are continuous and nonnegative, and

$$0 \leq h(x) \leq H(b - x) \tag{10}$$

for some constant $H \geq 0$.

Existence theorem

Under the assumptions (9), (10) **our** one-dimensional one-phase Stefan problem has at least one solution.

Proof of the existence theorem

$$\Sigma(A) = \{s \in C[0,T] | s(0) = b, 0 \leq \frac{s(t') - s(t'')}{t' - t''} \leq A, 0 \leq t' \leq t'' \leq T\}.$$

$$F : s \in \Sigma \to \sigma, \text{ where } \dot{\sigma} = -u_x(s(t), t), \sigma(0) = b.$$

Schauder's fixed point theorem

- Suppose Σ is non-empty, compact and convex;
- Suppose $F : \Sigma \to \Sigma$ is continuous.

Then F has a fixed point in Σ (i.e. $\exists s \in \Sigma : F(s) = s$).

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The plan of the proof

1. Σ is compact and convex;
2. $F(\Sigma) \subset \Sigma$;
3. F is continuous.

$$0 \leq u(x,t) \leq A(s(t)-x), 0 \leq x \leq s(t).$$

$$A := \max(H, \frac{1}{b} \sup_{t \in (0,T)} (f(t))).$$

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Continuity of the mapping

F is continuous

$\|\sigma_1 - \sigma_2\| \leq C\|s_1 - s_2\|$,where $\|\cdot\|$ denotes to the sup in $(0, T)$.

- For $s_1, s_2 \in \Sigma(s_1 < s_2)$ we consider u_1, u_2. $u_x(s(t), t) = -\dot{\sigma}(t)$.
- GHB:

$$\int_0^t s_i(\tau)\dot{\sigma}_i(\tau)d\tau - \int_0^{s_i(t)} u_i(x,t)dx + \int_0^b xh(x)dx - \int_0^t f(\tau)d\tau = 0, i = 1, 2.$$

- By substraction we get

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$$\int_0^{s_1(t)} (u_1(x,t) - u_2(x,t))dx = s_1(t)(\sigma_1(t) - \sigma_2(t)) - \int_0^t \dot{s}_1(\tau)(\sigma_1(\tau) - \sigma_2(\tau))d\tau.$$

Continuity of the mapping

F is continuous

$\|\sigma_1 - \sigma_2\| \leq C\|s_1 - s_2\|$,where $\|\cdot\|$ denotes to the sup in $(0, T)$.

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Continuity of the mapping

$$s_1(t)(\sigma_1(t)-\sigma_2(t)) = \int_0^t \dot{s}_1(\tau)(\sigma_1(\tau)-\sigma_2(\tau))d\tau - \int_0^t (s_1(\tau)-s_2(\tau))\dot{\sigma}_2(\tau)d\tau +$$

$$\int_0^{s_1(t)} (u_1(x,t)-u_2(x,t))dx - \int_{s_1(t)}^{s_2(t)} u_2(x,t)dx.$$

$$\int_0^t \dot{s}_1(\tau)(\sigma_1(\tau)-\sigma_2(\tau))d\tau \le A\int_0^t |\sigma_1(\tau)-\sigma_2(\tau)|d\tau \quad (11)$$

$$-\int_0^t (s_1(\tau)-s_2(\tau))\dot{\sigma}_2(\tau)d\tau \le A\int_0^t |s_1(\tau)-s_2(\tau)|d\tau \quad (12)$$

$$\int_0^{s_1(t)} (u_1(x,t)-u_2(x,t))dx \le s_1(t)\|s_1-s_2\| \quad (13)$$

$$-\int_{s_1(t)}^{s_2(t)} u_2(x,t)dx \le \frac{1}{2}A|s_1^2(t)-s_2^2(t)| \quad (14)$$

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- $s_1(t) \leq b + AT$, $\frac{1}{2}(s_1(t) + s_2(t)) \leq b + AT \Rightarrow$

$$|\sigma_1(t) - \sigma_2(t)| \leq \frac{A}{b} \int_0^t |\sigma_1(\tau) - \sigma_2(\tau)| d\tau + \frac{1}{b}[AT + b + AT + A(b + AT)]\|s_1 - s_2\|.$$

- Gronwall's inequality $\Rightarrow$ continuity ($\|\sigma_1 - \sigma_2\| \leq C\|s_1 - s_2\|$).

Gronwall's inequality

If for $t_0 \leq t \leq t_1$, $\phi(t)$ and $\psi(t)$ are continuous functions such that $\psi(t) \geq 0$ and

$$\phi(t) \leq K + \int_{t_0}^t \psi(s)\phi(s) ds, \; t_0 \leq t \leq t_1, \; K \text{ - const, then}$$

$$\phi(t) \leq K \exp(\int_{t_0}^t \phi(s) ds), \text{on } t_0 \leq t \leq t_1.$$

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For further reading

A. Fasano
Mathematical model of some diffusive processes with free boundaries
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C. Elliot and J. Ockendon
Weak and variational methods for moving boundary problems
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Questions?

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