Balance laws on domains with moving interfaces. The enthalpy method for the ice melting problem.

Godwin Kakuba

12th March 2008

Outline

Outline

Introduction

Motivation for the enthalpy method:

- enthalpy describes the state of the system completely as opposed to temperature

Consider a domain $\Omega(t)$ moving with a velocity U. On $\Omega(t)$ we want to compute

$$\frac{d}{dt}\int_{\Omega(t)} C(x,t)\,\mathrm{d}x\,.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} C(x,t)dx = \int_{\Omega(t)} \left(\frac{\partial C}{\partial t} + div(CU)\right)\mathrm{d}x$$

Introduction

Motivation for the enthalpy method:

- enthalpy describes the state of the system completely as opposed to temperature

Consider a domain $\Omega(t)$ moving with a velocity U. On $\Omega(t)$ we want to compute

$$\frac{d}{dt}\int_{\Omega(t)} C(x,t)\,\mathrm{d}x.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} C(x,t)dx = \int_{\Omega(t)} \left(\frac{\partial C}{\partial t} + div(CU)\right)\mathrm{d}x$$

Introduction

Motivation for the enthalpy method:

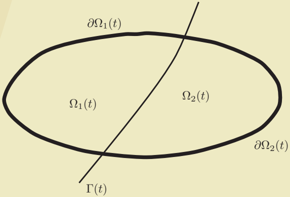
- enthalpy describes the state of the system completely as opposed to temperature

Consider a domain $\Omega(t)$ moving with a velocity U. On $\Omega(t)$ we want to compute

$$\frac{d}{dt}\int_{\Omega(t)} C(x,t)\,\mathrm{d}x.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} C(x,t)dx = \int\limits_{\Omega(t)} \left(\frac{\partial C}{\partial t} + div(CU)\right)\,\mathrm{d}x$$

So if we consider a domain with an interface $\Gamma(t)$



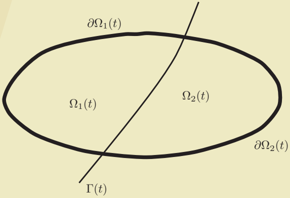
$\Omega(t) = \Omega_1(t) \cup \Omega_2(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_i(t)} C(x,t)\,\mathrm{d}x = \int\limits_{\Omega_i(t)} \frac{\partial C}{\partial t}\mathrm{d}x + \int_{\partial\Omega_i(t)} CU\cdot\mathbf{n}\,\mathrm{d}\sigma + \int_{\Gamma(t)} CW\cdot\mathbf{n}'\,\mathrm{d}\sigma$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} C(x,t)\,\mathrm{d}x = \int\limits_{\Omega_1(t)\cup\Omega_2(t)} \left(\frac{\partial C}{\partial t} + div(CU)\right)\mathrm{d}x + \int_{\Gamma(t)} [CV]\cdot\mathbf{n}'\,\mathrm{d}\sigma$$

$V := U - W$ is the relative velocity of the fluid with respect to $\Gamma(t)$.

So if we consider a domain with an interface $\Gamma(t)$



$\Omega(t) = \Omega_1(t) \cup \Omega_2(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_i(t)} C(x,t)\,\mathrm{d}x = \int_{\Omega_i(t)} \frac{\partial C}{\partial t}\mathrm{d}x + \int_{\partial\Omega_i(t)} CU\cdot\mathbf{n}\,\mathrm{d}\sigma + \int_{\Gamma(t)} CW\cdot\mathbf{n}'\,\mathrm{d}\sigma$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} C(x,t)\,\mathrm{d}x = \int_{\Omega_1(t)\cup\Omega_2(t)} \left(\frac{\partial C}{\partial t} + div(CU)\right)\,\mathrm{d}x + \int_{\Gamma(t)} [CV]\cdot\mathbf{n}'\,\mathrm{d}\sigma$$

$V := U - W$ is the relative velocity of the fluid with respect to $\Gamma(t)$.

Rankine-Hugoniot relations

Suppose

$$\frac{d}{dt}\int_{\Omega_i(t)} C \,\mathrm{d}x = \int_{\Omega_i(t)} f \,\mathrm{d}x \quad \text{for } i = 1,2;$$

$$\frac{\partial C}{\partial t} + div(CU) = f \quad \text{in} \quad \Omega(t)$$

$$\int_{\Gamma(t)} [CV]\cdot \mathbf{n}' \,\mathrm{d}\sigma = 0 \quad \text{in } \Gamma(t)$$

$$[CV]\cdot \mathbf{n}' = 0 \quad \text{on} \quad \Gamma(t)$$

Example: conservation of mass $C = \rho$

$$\rho_2 v_2 = \rho_1 v_1, \quad v = V \cdot \mathbf{n}'.$$

Balance of enthalpy

C is the enthalpy H;

$$\frac{d}{dt}\int_{\Omega_i(t)} H_i(x,t)\mathrm{d}x = \int_{\Omega_i(t)} f\mathrm{d}x + \int_{\partial\Omega_i(t)\cup\Gamma(t)} q_{H_i}\cdot\mathbf{n}\,\mathrm{d}\sigma$$

make use of Fourier's law,

we consider $U = 0$,

and arrive at

$$\int_{\Omega(t)} \left(\frac{\partial H}{\partial t}(x,t) - div(k\nabla T) - f\right)\mathrm{d}x + \int_{\Gamma(t)} \left([HW]\cdot\mathbf{n}' - [q_H\cdot\mathbf{n}']\right)\mathrm{d}\sigma = 0$$

Thus

$$\frac{\partial H}{\partial t}(x,t) - div(k\nabla T) - f = 0$$

$$[HW]\cdot\mathbf{n}' - [q_H\cdot\mathbf{n}'] = 0, \quad \text{the Stefan condition}$$

$[H] = H_2 - H_1 = L$ is the latent heat of melting

Outline

Consider

$$\frac{\partial H}{\partial t} - div(k \nabla T) = 0.$$

We introduce the function θ as

$$\theta = \begin{cases} \int_{T_m}^{T} k_l(\tau)\,\mathrm{d}\tau, & T \geq T_m \\ \int_{T_m}^{T} k_s(\tau)\,\mathrm{d}\tau, & T < T_m \end{cases}$$

So we have the field equation

$$\frac{\partial H}{\partial t} - \Delta\theta = 0 \text{ in } Q_s \cup Q_l$$

Definition of the domain:

$$
\begin{aligned}
Q_T &= \Omega \times (0, T) \\
Q_l &= Q_T \cap \{H > L\} \\
Q_s &= Q_T \cap \{H < 0\} \\
\Omega_l(t) &= Q_l \cap \{t\} \\
\Omega_s(t) &= Q_s \cap \{t\} \\
\Gamma_l & \quad \text{external boundary of } Q_l \\
\Gamma_s & \quad \text{external boundary of } Q_s \\
\Gamma_0 & \quad \text{interface in } \mathbb{R}^{n+1} \\
\Gamma(t) &= \Gamma_0 \cap \{t\} \\
\mathbf{n} &= \text{normal to } \Gamma(t) \text{ pointing towards } \Omega_l(t)
\end{aligned}
$$

$$H(x, 0) = H_0(x) \in \Omega_{l0} \cup \Omega_{s0}$$

$$T|_{\Gamma_l} = f_l(t) > T_m, \quad T|_{\Gamma_s} = f_s(t) < T_m$$

$$LV \cdot \mathbf{n} = -\frac{\partial \theta_l}{\partial n} + \frac{\partial \theta_s}{\partial \theta_n}$$

Definition of the domain:

$$
\begin{aligned}
Q_T &= \Omega \times (0, T) \\
Q_l &= Q_T \cap \{H > L\} \\
Q_s &= Q_T \cap \{H < 0\} \\
\Omega_l(t) &= Q_l \cap \{t\} \\
\Omega_s(t) &= Q_s \cap \{t\} \\
\Gamma_l & \quad \text{external boundary of } Q_l \\
\Gamma_s & \quad \text{external boundary of } Q_s \\
\Gamma_0 & \quad \text{interface in } \mathbb{R}^{n+1} \\
\Gamma(t) &= \Gamma_0 \cap \{t\} \\
\mathbf{n} &= \text{normal to } \Gamma(t) \text{ pointing towards } \Omega_l(t)
\end{aligned}
$$

$$H(x, 0) = H_0(x) \in \Omega_{l0} \cup \Omega_{s0}$$

$$T|_{\Gamma_l} - f_l(t) > T_m, \quad T|_{\Gamma_s} = f_s(t) < T_m$$

$$L\mathbf{V} \cdot \mathbf{n} = -\frac{\partial \theta_l}{\partial n} + \frac{\partial \theta_s}{\partial \theta_n}$$

Weak formulation

- Introduce the set of test functions

$$\mathbb{V} = \{\varphi \in \mathcal{W}^{2,1}(Q_T) |\ \varphi = 0 \text{ on } \Gamma_l \cup \Gamma_s \cup \Omega \times \{T\}\}$$

- multiply by φ and use partial integration

In Q_s

$$\int_{Q_s} \left(\frac{\partial H}{\partial t} - \Delta\theta\right) \varphi dxdt = -\int_{Q_s} (H\frac{\partial \varphi}{\partial t} + \theta\Delta\varphi)dxdt - \int_{\Omega_{0s}} H_0\varphi(x,0)dx + \int_{\Gamma_s} \theta(f_s)\frac{\partial\varphi}{\partial n_s} d\sigma dt - \int_0^T \int_{\Gamma(t)^-} \varphi\frac{\partial\theta_s}{\partial n} d\sigma dt$$

In Q_l

$$\int_{Q_l} \left(\frac{\partial H}{\partial t} - \Delta\theta\right) \varphi dxdt = -\int_{Q_l} (H\frac{\partial \varphi}{\partial t} + \theta\Delta\varphi)dxdt - \int_{\Omega_{0l}} H_0\varphi(x,0)dx + \int_{\Gamma_l} \theta(f_l)\frac{\partial\varphi}{\partial n_l} d\sigma dt - \int_0^T \int_{\Gamma(t)^+} \varphi\left[\frac{\partial\theta_l}{\partial n} + V_n L\right] d\sigma dt$$

Weak formulation

- Introduce the set of test functions

$$\mathbb{V} = \{\varphi \in \mathcal{W}^{2,1}(Q_T) | \ \varphi = 0 \text{ on } \Gamma_l \cup \Gamma_s \cup \Omega \times \{T\}\}$$

- multiply by φ and use partial integration

In Q_s

$$\int_{Q_s} \left(\frac{\partial H}{\partial t} - \Delta\theta \right) \varphi dxdt = -\int_{Q_s} (H \frac{\partial \varphi}{\partial t} + \theta \Delta \varphi) dxdt - \int_{\Omega_{0s}} H_0 \varphi(x,0) dx$$
$$+ \int_{\Gamma_s} \theta(f_s) \frac{\partial \varphi}{\partial n_s} d\sigma dt - \int_0^T \int_{\Gamma(t)-} \varphi \frac{\partial \theta_s}{\partial n} d\sigma dt$$

In Q_l

$$\int_{Q_l} \left(\frac{\partial H}{\partial t} - \Delta\theta \right) \varphi dxdt = -\int_{Q_l} (H \frac{\partial \varphi}{\partial t} + \theta \Delta \varphi) dxdt - \int_{\Omega_{0l}} H_0 \varphi(x,0) dx +$$
$$\int_{\Gamma_l} \theta(f_l) \frac{\partial \varphi}{\partial n_l} d\sigma dt - \int_0^T \int_{\Gamma(t)+} \varphi \left[\frac{\partial \theta_l}{\partial n} + V_n L \right] d\sigma dt$$

Weak formulation

Add and use the Stefan condition to get:

$$\int_{Q_T} \left(H\frac{\partial \varphi}{\partial t} + \theta \Delta \varphi \right) dxdt = \int_{\Gamma_s} \theta(f_s) \frac{\partial \varphi}{\partial n_s} d\sigma dt + \int_{\Gamma_l} \theta(f_l) \frac{\partial \varphi}{\partial n_l} d\sigma dt - \int_{\Omega_0} H_0 \varphi(x,0) dx \tag{1}$$

Definition

A weak solution of the Stefan problem is a function $H \in L^\infty(Q_T)$ satisfying this integral for all $\varphi \in \mathbb{V}$.

Existence

assumptions

- Suppose the boundary of Ω_0, and consequently Γ_s, Γ_l, are smooth
- f_s, f_l are continuously differentiable and bounded away from zero in $\bar{\Gamma}_s, \bar{\Gamma}_l$
- for $t = 0, f_s$ and f_l match the initial data and $H_0 \in C^1(\bar{\Omega}_0)$
- k is twice continuously differentiable

Under these assumptions, there is at least one weak solution.
Proof.

- Regularization
- Consider the problems

$$\begin{cases} H_n'(\theta_n)\frac{\partial\theta_n}{\partial t} - \Delta\theta_n = 0 \ \text{ in } \ Q_T \\ \theta_n|_{\Gamma_s} = \theta(f_s), \ \ \theta_n|_{\Gamma_l} = \theta(f_l) \\ \theta_n(x,0) = \Theta_n(H_0(x)) \end{cases}$$

Existence

- From classical literature, there is a unique solution to these problems for each n
- $\left.\frac{\partial\theta_n}{\partial n_s}\right|_{\Gamma_s}, \left.\frac{\partial\theta_n}{\partial n_l}\right|_{\Gamma_l}$ are uniformly bounded

Lemma

There is a constant C independent of n and determined by the data such that

$$\omega \int_{Q_t} \left(\frac{\partial\theta_n}{\partial t}\right)^2 dx d\tau + \int_{\Omega\times\{t\}} \frac{1}{2}|\nabla\theta_n|^2 \, \mathrm{d}x \leq C dx, \quad \forall t \in (0, T)$$

Consequently

- the sequence $\{\theta_n\}$ is weakly compact in $\mathcal{H}^1(Q_T)$.
  $\Rightarrow$
- there is a subsequence $\{\theta_{n'}\}$ converging almost uniformly to θ in Q_T

Existence

Consider $\{H_{n'}(\theta_{n'})\}$, select a subsequence which is weakly convergent in $L^2(Q_T)$ to $\tilde{H}(x,t) \in L^\infty$

- We may interpret $\tilde{H}(x,t)$ as $H(\theta(x,t))$ a.e. in Q_T

So, take the weak version:

$$\int_{Q_T} \{H_n(\theta_n)\frac{\partial \varphi}{\partial t} + \theta_n \Delta\varphi\} dxdt = \int_{\Gamma_s} \theta(f_s)\frac{\partial \varphi}{\partial n_s} d\sigma dt + \int_{\Gamma_l} \theta(f_l)\frac{\partial \varphi}{\partial n_l} d\sigma dt - \int_{\Omega_0} H_0 \varphi(x) dx, \ \forall \varphi \in \mathbb{V} \tag{2}$$

Exploit the limits $\theta_n \xrightarrow{L^2} \theta,\ H_n(\theta_n) \xrightharpoonup{L^2} \tilde{H}$ and the properties of $\tilde{H}$ to conclude the existence of the weak solution.

Summary

- R. Teaman & A. Miranville, *Mathematical Modeling in Continuum Mechanics, Cambridge University Press, 2000*
- Antonio Fasano, *Mathematical Models of some Diffusive Processes with Free Boundaries*
- S.J.L. van Eindhoven, *Mathematical Models based on free boundary problems*