Finite Element Approximation of the Enthalpy Method for the Stefan Problem

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March 26, 2008

Outline

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Recapitulation

Previous seminar:

- Balance equations
  - ✓ Rankine-Hugoniot relations
  - ✓ Stefan conditions for melting of ice

- The enthalpy method for the Stefan problem
  - ✓ Weak formulation
  - ✓ Existence of a weak solution

Nomenclature

- Ω: polygonal domain in $\mathbb{R}^2$
- $Q = \Omega \times (0, t_{\mathrm{end}})$
- h: enthalpy
- $\theta = \int_{T_\infty}^{T} \kappa(\tau)\mathrm{d}\tau$,
  - T: temperature
  - T_∞: constant temperature on $\partial\Omega$
  - κ: heat conductivity
- $\theta_m = \theta(T_m)$, with T_m melting point
- f: body heating

Enthalpy Method for the Stefan Problem

$$\begin{cases} \frac{\partial h}{\partial t} - \Delta\theta = f, & \text{in } Q \\ h(\mathbf{x}, t) \in \beta\big(\theta(\mathbf{x}, t)\big), & \text{for } (\mathbf{x}, t) \in Q \\ \theta(\mathbf{x}, t) \ = 0, & \text{on } \partial\Omega \\ h(\mathbf{x}, 0) = h_0(\mathbf{x}), & \text{in } \Omega \end{cases}$$

where

$$\beta(\theta) = \begin{cases} \alpha_1(\theta - \theta_m), & \theta < \theta_m, \\ \big[0, L\big], & \theta = \theta_m, \\ \alpha_2(\theta - \theta_m) + L, & \theta > \theta_m \end{cases}$$

Weak Formulation

$$(\frac{\partial h}{\partial t}, \omega) + a(\theta, \omega) = (f, \omega), \qquad \text{for all } \omega \in H_0^1(\Omega)$$

where

$$a(\theta, \omega) = \int_\Omega \nabla\theta \cdot \nabla\omega \mathrm{d}\mathbf{x}$$

Motivation:

- no smooth functions required
- implicit definition of moving interface
- efficient numerical methods based on weak formulation

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Discretisation by Finite Elements

- Cover Ω by triangular mesh $\mathcal{T}_\ell$
  - ✓ ℓ: longest triangle edge
  - ✓ angles bounded below
  - ✓ aspect ratio bounded above
- Define linear vector space $V_\ell := \left\{ v = \sum_{j \in J} v_j \, \varphi_j \right\}$
  - ✓ $\varphi_j,\ j \in J$: piecewise linear basis functions
  - ✓ J: set of nodes

Discrete L^2 inner products in V_ℓ

$$
\begin{aligned}
(u, v)_0 &= \underset{\sim}{\mathbf{u}}^T \mathbf{M} \underset{\sim}{\mathbf{v}} = \frac{1}{3} \sum_{j \in J} w_j u_j v_j \\
(u, v)_1 &= \underset{\sim}{\mathbf{u}}^T \mathbf{A} \underset{\sim}{\mathbf{v}} = \sum_{i \in J} \sum_{j \in J} u_i a(\varphi_i, \varphi_j) v_j
\end{aligned}
$$

- **M**: lumped mass matrix ($M_{jj} = \frac{1}{3} w_j$)
- w_j: weights
- **A**: discrete Laplacian ($A_{ij} = a(\varphi_i, \varphi_j)$)

Discrete Inverse Norm inequality

$$\|v\|_1 \leq S(\ell)\|v\|_0, \qquad \text{for all } v \in H_0^1(\Omega) \cap H^2(\Omega)$$

where

$$S(\ell) = \widetilde{C}\frac{\ell}{\min_{T \in \mathcal{T}_\ell} A(T)}$$

with $A(T)$ area of triangle T

Discrete Problem

Find vectors $\underset{\sim}{\mathbf{h}}^k$, $\underset{\sim}{\boldsymbol{\theta}}^k$ with nodal values at $t = k\Delta t$, such that

$$\begin{cases} \delta \underset{\sim}{\mathbf{h}}^k + \mathbf{M}^{-1}\mathbf{A}\underset{\sim}{\boldsymbol{\theta}}^{k+\gamma} = \underset{\sim}{\mathbf{f}}^{k+\gamma} \\ h_j^{k+1} \in \beta\left(\theta_j^{k+1}\right). \end{cases}$$

where

$$\delta \underset{\sim}{\mathbf{h}}^k = \frac{\underset{\sim}{\mathbf{h}}^{k+1} - \underset{\sim}{\mathbf{h}}^k}{\Delta t}$$

$$\underset{\sim}{\boldsymbol{\theta}}^{k+\gamma} = \gamma\, \underset{\sim}{\boldsymbol{\theta}}^{k+1} + (1-\gamma)\underset{\sim}{\boldsymbol{\theta}}^k, \quad 0 \le \gamma \le 1,$$

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Stability Estimate

Theorem

If

$$\left(1 - 2\gamma\right)\Delta t\left(S(\ell)\right)^2 < 2\alpha,$$

then there is a constant C (independent of $\ell, \Delta t$) such that

$$\alpha \sum_{j=0}^{k} \Delta t \|\delta\theta^j\|_0^2 + \|\theta^{k+1}\|_1^2 \leq C\left(\|\theta^0\|_1^2 + \sum_{j=0}^{k} \Delta t \|f^{j+\gamma}\|_0^2\right)$$

$$(u,v)_0 = \underset{\sim}{\mathbf{u}}^T \mathbf{M} \underset{\sim}{\mathbf{v}} = \frac{1}{3} \sum_{j \in J} w_j u_j v_j, \qquad (u,v)_1 = \underset{\sim}{\mathbf{u}}^T \mathbf{A} \underset{\sim}{\mathbf{v}} = \sum_{i \in J} \sum_{j \in J} u_i a(\varphi_i, \varphi_j) v_j$$

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1. Introduction
2. Finite Element Discretisation
3. Stability
4. **Existence and Uniqueness**
5. Summary

Implicit Scheme

Each time step:

- solve system of nonlinear algebraic equations

$$\underset{\sim}{\mathbf{h}} + \mathbf{C}\underset{\sim}{\theta} = \underset{\sim}{\mathbf{b}}, \qquad h_j \in \beta(\theta_j)$$

  - **C**: symmetric positive definite matrix
  - β: monotone multifunction

Existence and Uniqueness?

Concept of proof:

solving implicit scheme $\Longleftrightarrow$ finding minimum of non-differentiable functional

Generalised Concept of Derivative

Consider functional $\Phi : H \to \mathbb{R}$, with H Hilbert space

- Convex functional:

$$\Phi(\lambda x_1 + (1-\lambda)x_2) \;\leq\; \lambda\Phi(x_1) + (1-\lambda)\Phi(x_2), \qquad \lambda \in [0,1]$$

- Subdifferential:

$$\partial\Phi(x) \;=\; \{z \in H \,|\, \Phi(y) - \Phi(x) \geq (z, y-x) \text{ for all } y \in H\}$$

  - ✓ x minimum of $\Phi \iff 0 \in \partial\Phi(x)$
  - ✓ Φ differentiable in $x \iff \partial\Phi(x)$ consists of derivative only

Minimisation Problem

- Define $\phi : \mathbb{R} \to \mathbb{R}$, such that $\partial\phi = \beta$:

$$\phi(v) = \frac{1}{2}\alpha_1(v-\theta_m)_-^2 + \frac{1}{2}\alpha_2(v-\theta_m)_+^2 + L(v-\theta_m)_+$$

$$v_+ = \begin{cases} v, & \text{if } v > 0 \\ 0, & \text{if } v < 0 \end{cases}, \quad v_- = \begin{cases} v, & \text{if } v < 0 \\ 0, & \text{if } v > 0 \end{cases}$$

- Find minimum of

$$\Phi(\underset{\sim}{\mathbf{v}}) = \sum_{j=1}^{J} \phi(v_j) + \frac{1}{2}\underset{\sim}{\mathbf{v}}^T \mathbf{C}\underset{\sim}{\mathbf{v}} - \underset{\sim}{\mathbf{b}}^T \underset{\sim}{\mathbf{v}}$$

Proof of Existence and Uniqueness

$$\Phi(\underset{\sim}{\mathbf{v}}) = \sum_{j=1}^{J} \phi(v_j) + \frac{1}{2}\underset{\sim}{\mathbf{v}}^T \mathbf{C} \underset{\sim}{\mathbf{v}} - \underset{\sim}{\mathbf{b}}^T \underset{\sim}{\mathbf{v}}$$

Lemma:

Let Φ be strictly convex and continuous in Hilbert space H with $\Phi(v) \to \infty$ as $\|v\| \to \infty$, then there exists a unique minimum of Φ characterised by $0 \in \partial\Phi(\theta)$

- Φ satisfies hypotheses in lemma
- $\underset{\sim}{\mathbf{v}} \in \partial\Phi(\underset{\sim}{\theta}) \iff v_i \in \beta(\theta_i) + \left(\mathbf{C}\underset{\sim}{\theta} - \underset{\sim}{\mathbf{b}}\right)_i$
- $\underset{\sim}{\theta}$ minimum of $\Phi \iff \underset{\sim}{\mathbf{h}} + \mathbf{C}\underset{\sim}{\theta} - \underset{\sim}{\mathbf{b}} = \mathbf{0}$

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- Approximation of the enthalpy method for the Stefan problem by finite elements
- Proof of a stability estimate for the discrete problem
- Proof of existence and uniqueness for implicit schemes

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References

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