The Obstacle Problem

A Variational Inequalities Approach

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5th Talk on Free and Moving Boundary Value Problems

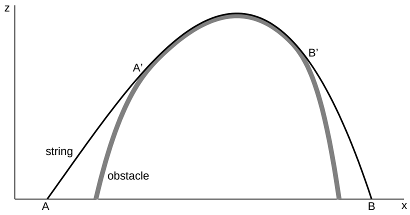
16th April 2008

The Obstacle Problem

Outline

- Obstacle Problem in 1D
  - $\Rightarrow$ Free boundary value problem $(*)$
  - $\Rightarrow$ Variational problem $(**)$
  - $\Rightarrow$ Variational inequality
- Obstacle problem in 2D
  - $\Rightarrow$ Equivalence between $(*)$ and $(**)$
  - $\Rightarrow$ Abstract elliptic variational inequalities
  - $\Rightarrow$ Existence and uniqueness of the obstacle problem

The Obstacle Problem for a String



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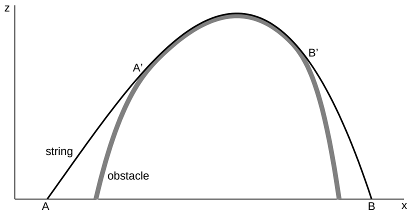

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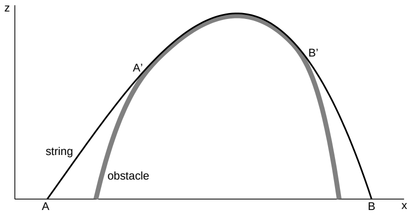

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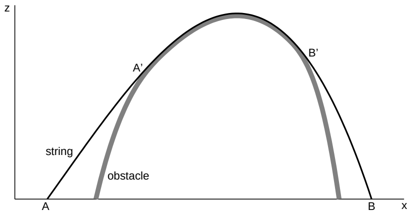

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$$u \geq \psi, \quad \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \leq 0, \quad \text{on } AB.$$

Variational Formulation

- For a fixed domain the variational problem is

$$\min_{v} \left(\int_{A}^{A'} + \int_{B}^{B'} \right) \left(\frac{\mathrm{d}v}{\mathrm{d}x} \right)^2 \, \mathrm{d}x,$$

for v suitably smooth and prescribed at A, A′, B, B′.

- For the moving domain, we consider

$$\min_{v \geq \psi} \int_{A}^{B} \left(\frac{\mathrm{d}v}{\mathrm{d}x} \right)^2 \, \mathrm{d}x,$$

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- For all $v \geq \psi$ and suitably smooth, we require

$$a(u, v-u) = \int_A^B \frac{\mathrm{d}u}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x}\right)\,\mathrm{d}x \geq 0.$$

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$$= \int_A^B \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x + 2\lambda\int_A^B \frac{\mathrm{d}u}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x + O\left(\lambda^2\right).$$

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Eliminating the red terms, we find

$$\int_A^B \frac{\mathrm{d}u}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x}\right)\,\mathrm{d}x + O(\lambda) \geq 0, \qquad \lambda \ll 1.$$

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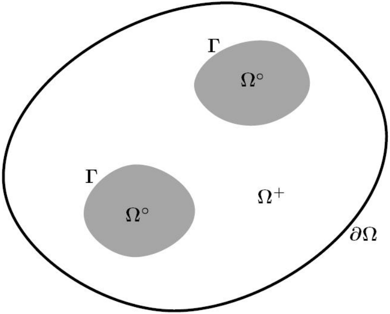
$$= -\int_{A'}^{B'} \frac{d^2\psi}{dx^2}(v-\psi)\,dx \geq 0.$$

- If there is a production term f, then we have

$$a(u, v-u) \geq (f, v-u),$$

for all suitable $v \geq \psi$.

The Obstacle Problem in $\mathbb{R}^2$



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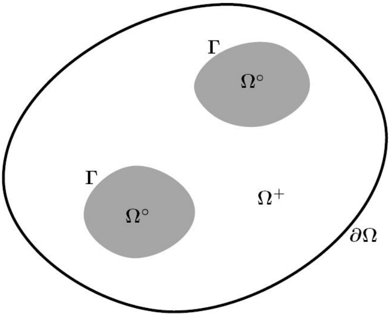

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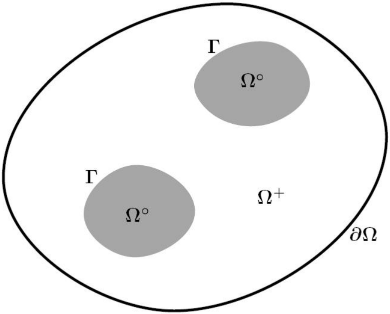

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u \geq \psi, \quad -\Delta u \geq f, \quad \text{in } \Omega.
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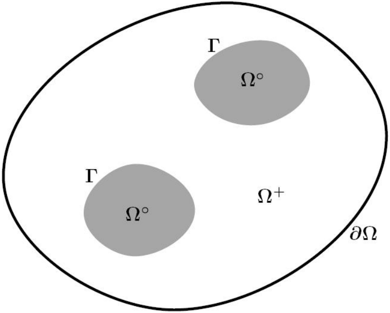

Abstract Variational Inequalities

Let V be a real Hilbert space and K a closed, convex, non-empty subset of V.

Let $a : V \times V \to \mathbb{R}$ be a continuous, coercive, bilinear form on V, so that $\exists \alpha, \beta > 0$

$$\begin{aligned} |a(u,v)| &\le \beta \|u\|\|v\|, && \text{for all } u, v \in V,\\ a(v,v) &\ge \alpha \|v\|^2, && \text{for all } v \in V,\\ \Rightarrow \quad \alpha\|v\|^2 \le a(v,v) &\le \beta\|v\|^2, && \text{for all } v \in V. \end{aligned}$$

Let $\ell : V \to \mathbb{R}$ be a continuous, linear mapping so that $\exists M > 0$

$$|\ell(v)| \le M\|v\|, \quad \text{for all } v \in V.$$

Abstract Variational Inequalities

We consider the variational inequality

(P1) *Find* $u \in K$ *such that* $a(u, v - u) \geq \ell(v - u)$ *for all* $v \in K$*.*

For symmetric a, we also consider the minimization problem

(P2) *Find* $u \in K$ *such that* $E(u) = \min_K E(v)$,

where $E(v) = \frac{1}{2}a(v, v) - \ell(v)$.

Theorem I: existence and uniqueness

There exists a unique solution to (P1). If $a(\cdot, \cdot)$ is symmetric, then (P1) and the (P2) are equivalent.

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For the obstacle problem we have

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\mathrm{d}y, \qquad \ell(v) = \int_\Omega fv \,\mathrm{d}x\mathrm{d}y,$$

$$E(v) = \int_\Omega \left[\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2 - fv\right] \mathrm{d}x\mathrm{d}y.$$

We take

$$V = H_0^1(\Omega), \qquad K = \{v \in V : v \geq \psi \text{ a.e. in } \Omega\},$$

and suppose $f \in L^2(\Omega)$, $\psi \in H^2(\Omega) \cap C^0(\overline{\Omega})$.

Abstract Variational Inequalities

Theorem II: existence and uniqueness for the obstacle problem

There exists a unique $u \in K$ such that

(i) $E(u) \leq E(v) \qquad \text{for all } v \in K,$

or equivalently

(ii) $\int_\Omega \nabla u \cdot \nabla (v-u) \, \mathrm{d}x\mathrm{d}y \geq \int_\Omega f(v-u) \, \mathrm{d}x\mathrm{d}y \qquad \text{for all } v \in K.$

Theorem III: regularity

Let $\partial\Omega$ be smooth. If $f \in L^p(\Omega)$ and $\psi \in W^{2,p}(\Omega)$ for $p \in (1,\infty)$ then the solution u to (i) or (ii) lies in $W^{2,p}(\Omega)$.

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Theorem IV: principal theorem

(a) Let $f \in L^p(\Omega)$ and $\psi \in W^{2,p}(\Omega)$ for some $p > 2$. Then the solution of the variational inequality (ii) solves the free BVP.

(b) Let $f \in L^2(\Omega)$ and $\psi \in H^2(\Omega)$. Suppose $\{u, \Gamma\}$ is the solution of the free BVP such that $u \in H_0^1 \cap H^2(\Omega) \cap C^1(\overline{\Omega})$ and Γ is smooth. Then u solves the variational inequality (ii) .

Possible extensions

- Variable coefficients
- $K = \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega \text{ and } v \geq \psi \text{ in } \Omega\}$.
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Summary

- Obstacle problem in 1D and 2D
- Existence and uniqueness of the variational inequality
- Equivalence of the minimization problem and variational inequality
- Equivalence of free boundary value problem and variational inequality

Further reading

C. Elliot and J. Ockendon
Weak and variational methods for moving boundary problems
1982.