# The Obstacle Problem

## A Variational Inequalities Approach

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5th Talk on Free and Moving Boundary Value Problems

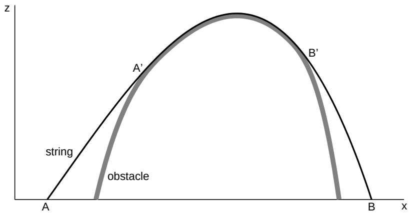
16th April 2008

# The Obstacle Problem

## Outline

- Obstacle Problem in 1D
  - $\Rightarrow$ Free boundary value problem $(*)$
  - $\Rightarrow$ Variational problem $(**)$
  - $\Rightarrow$ Variational inequality
- Obstacle problem in 2D
  - $\Rightarrow$ Equivalence between $(*)$ and $(**)$
  - $\Rightarrow$ Abstract elliptic variational inequalities
  - $\Rightarrow$ Existence and uniqueness of the obstacle problem

# The Obstacle Problem for a String



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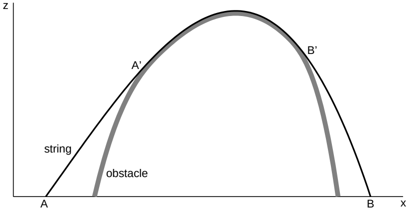

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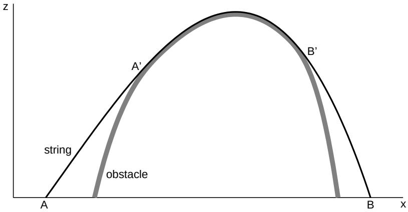

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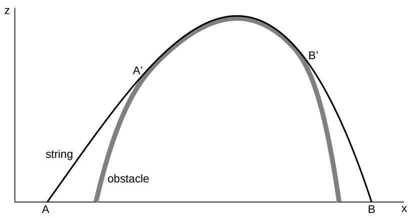


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$$u \geq \psi, \quad \frac{d^2u}{dx^2} \leq 0, \quad \text{on } AB.$$

# Variational Formulation

- For a fixed domain the variational problem is

$$\min_{v} \left( \int_{A}^{A'} + \int_{B}^{B'} \right) \left( \frac{\mathrm{d}v}{\mathrm{d}x} \right)^2 \, \mathrm{d}x,$$

for $v$ suitably smooth and prescribed at A, A′, B, B′.

- For the moving domain, we consider

$$\min_{v \geq \psi} \int_{A}^{B} \left( \frac{\mathrm{d}v}{\mathrm{d}x} \right)^2 \, \mathrm{d}x,$$

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# Variational Inequality

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$$a(u, v-u) = \int_A^B \frac{\mathrm{d}u}{\mathrm{d}x} \left( \frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x} \right) \, \mathrm{d}x \geq 0.$$

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$$= \int_A^B \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x + 2\lambda \int_A^B \frac{\mathrm{d}u}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x + O\left(\lambda^2\right).$$

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Eliminating the red terms, we find

$$\int_A^B \frac{\mathrm{d}u}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x + O(\lambda) \geq 0, \qquad \lambda \ll 1.$$

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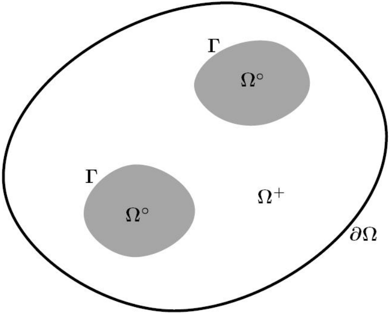
$$= -\int_{A'}^{B'} \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(v-\psi)\,\mathrm{d}x \geq 0.$$

- If there is a production term $f$, then we have

$$a(u, v-u) \geq (f, v-u),$$

for all suitable $v \geq \psi$.

# The Obstacle Problem in $\mathbb{R}^2$



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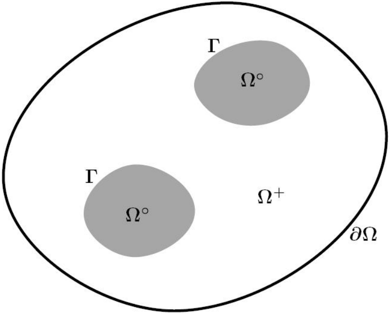

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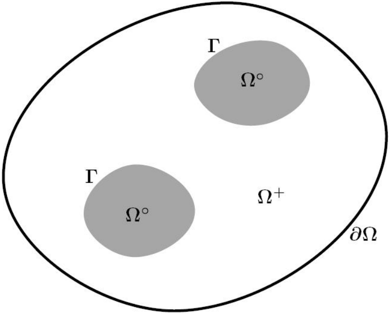

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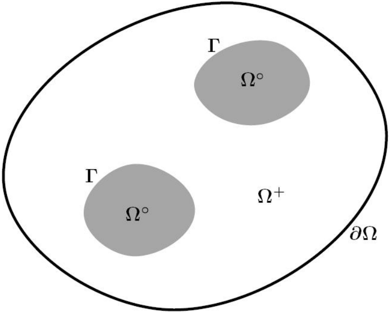
$$u \geq \psi, \quad -\Delta u \geq f, \quad \text{in } \Omega.$$



# Abstract Variational Inequalities

Let $V$ be a real Hilbert space and $K$ a closed, convex, non-empty subset of $V$.

Let $a : V \times V \to \mathbb{R}$ be a continuous, coercive, bilinear form on $V$, so that $\exists \alpha, \beta > 0$

$$
\begin{aligned}
|a(u,v)| &\leq \beta \|u\| \|v\|, && \text{for all } u, v \in V,\\
a(v,v) &\geq \alpha \|v\|^2, && \text{for all } v \in V,\\
\Rightarrow \quad \alpha \|v\|^2 &\leq a(v,v) \leq \beta \|v\|^2, && \text{for all } v \in V.
\end{aligned}
$$

Let $\ell : V \to \mathbb{R}$ be a continuous, linear mapping so that $\exists M > 0$

$$
|\ell(v)| \leq M \|v\|, \quad \text{for all } v \in V.
$$

# Abstract Variational Inequalities

We consider the variational inequality

*(P1) Find* $u \in K$ *such that* $a(u, v-u) \geq \ell(v-u)$ *for all* $v \in K$.

For symmetric $a$, we also consider the minimization problem

*(P2) Find* $u \in K$ *such that* $E(u) = \min_K E(v)$,

where $E(v) = \frac{1}{2}a(v,v) - \ell(v)$.

**Theorem I: existence and uniqueness**

There exists a unique solution to (P1). If $a(\cdot,\cdot)$ is symmetric, then (P1) and the (P2) are equivalent.

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# Abstract Variational Inequalities

For the obstacle problem we have

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y, \qquad \ell(v) = \int_\Omega fv \, \mathrm{d}x\mathrm{d}y,$$

$$E(v) = \int_\Omega \left[ \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial y} \right)^2 - fv \right] \mathrm{d}x\mathrm{d}y.$$

We take

$$V = H_0^1(\Omega), \qquad K = \{ v \in V : v \geq \psi \text{ a.e. in } \Omega \},$$

and suppose $f \in L^2(\Omega)$, $\psi \in H^2(\Omega) \cap C^0(\overline{\Omega})$.

# Abstract Variational Inequalities

**Theorem II: existence and uniqueness for the obstacle problem**

There exists a unique $u \in K$ such that

(i) $E(u) \leq E(v) \qquad \text{for all } v \in K,$

or equivalently

(ii) $\int_\Omega \nabla u \cdot \nabla(v-u)\,\mathrm{d}x\mathrm{d}y \geq \int_\Omega f(v-u)\,\mathrm{d}x\mathrm{d}y \qquad \text{for all } v \in K.$

**Theorem III: regularity**

Let $\partial\Omega$ be smooth. If $f \in L^p(\Omega)$ and $\psi \in W^{2,p}(\Omega)$ for $p \in (1,\infty)$ then the solution $u$ to (i) or (ii) lies in $W^{2,p}(\Omega)$.

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# Abstract Variational Inequalities

## Theorem IV: principal theorem

(a) Let $f \in L^p(\Omega)$ and $\psi \in W^{2,p}(\Omega)$ for some $p > 2$. Then the solution of the variational inequality (ii) solves the free BVP.

(b) Let $f \in L^2(\Omega)$ and $\psi \in H^2(\Omega)$. Suppose $\{u, \Gamma\}$ is the solution of the free BVP such that $u \in H_0^1 \cap H^2(\Omega) \cap C^1(\overline{\Omega})$ and $\Gamma$ is smooth. Then $u$ solves the variational inequality (ii) .

## Possible extensions

- Variable coefficients
- $K = \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega \text{ and } v \geq \psi \text{ in } \Omega\}$.
- Relaxed smoothness of the boundaries $\partial\Omega$ and $\Gamma$

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# Summary

- Obstacle problem in 1D and 2D
- Existence and uniqueness of the variational inequality
- Equivalence of the minimization problem and variational inequality
- Equivalence of free boundary value problem and variational inequality

# Further reading

C. Elliot and J. Ockendon
Weak and variational methods for moving boundary problems
1982.