Finite element approximation of the elliptic obstacle problem

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Outline

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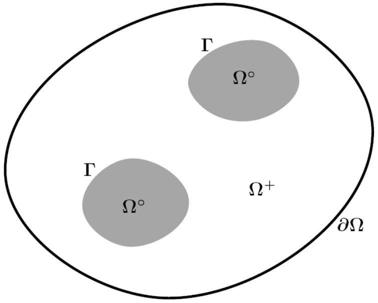
Problem Definition

Obstacle Problem

$$-\Delta u = f \quad \text{in } \Omega^+ \tag{1}$$
$$u = \Psi \quad \text{in } \Omega^0 \tag{2}$$
$$u = 0 \quad \text{on } \partial\Omega \tag{3}$$
$$u = \psi \,,\; \nabla u = \nabla\Psi \quad \text{on } \Gamma \tag{4}$$



Additional Inequalities

$$u \geq \Psi \quad \text{in } \Omega \tag{5}$$
$$-\Delta u \geq f \quad \text{in } \Omega \tag{6}$$

Weak Formulation

$$K = \{v \in H_0^1(\Omega) \mid v \geq \Psi \text{ a.e. in } \Omega\} \tag{7}$$

$$\Psi \in H^2(\Omega) \cap C^0(\bar{\Omega}) \tag{8}$$

$$\Psi < 0 \text{ on } \partial\Omega \tag{9}$$

$$f \in \mathbb{L}^2(\Omega) \tag{10}$$

Theorem

There exists a unique $u \in K$ such that

$$\int_\Omega \nabla u \cdot \nabla(v - u)dxdy \geq \int_\Omega f(v - u)dxdy \text{ for all } v \in K \tag{11}$$

and this u solves the obstacle problem.

Equivalent formulation

Constrained minimization problem

Find $u \in K$ such that it is the solution to

$$\min_{v \in K} \frac{1}{2} \int_{\Omega} \nabla v \cdot \nabla v \, \mathrm{d}x\mathrm{d}y - \int_{\Omega} f v \, \mathrm{d}x\mathrm{d}y \qquad (12)$$

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Discretization I

Let Ω be a convex polygon and $V^h =< \phi_j >$ the finite element space approximating $H_0^1(\Omega)$. Then

$$v(x) = \sum_{j \in J} v_j \phi_j(x) \tag{13}$$

The approximation of K:

$$K^h = \{\phi \in V^h \mid \phi \geq \Psi\ \} \tag{14}$$

Definition

For any continuous function v on Ω we define the interpolate v_I of v such that $v_I \in V^h$ and $v_I = v$ at each vertex of the triangulation of Ω.

Discretization II

Next we consider the weak formulation of the obstacle problem over K^h,

$$\int_\Omega \nabla u^h \cdot \nabla(v - u^h)\mathrm{d}x\mathrm{d}y \geq \int_\Omega f(v - u^h)\mathrm{d}x\mathrm{d}y \text{ for all } v \in K^h \tag{15}$$

(16)

Which we will write as

$$a(u^h, v - u^h) \geq (f, v - u^h) \;\text{ for all } v \in K^h \tag{17}$$

Discretization III

With equivalent minimization problem

$$\arg\min_{v \in K^h} J[v] \ , \ J[v] = \frac{1}{2} a(v, v) - (f, v) \ . \qquad (18)$$

Which can be expressed in matrix terms as

$$\arg\min_{\vec{v} \geq \vec{\Psi}} J[\vec{v}] \ , \ J[\vec{v}] = \frac{1}{2} \vec{v}^T A \vec{v} - \vec{f}^T \vec{v} \ , \qquad (19)$$

with the vectors $\vec{v}, \vec{\Psi}$ the nodal values and $A_{ij} = a(\phi_i, \phi_j)$ and $(\vec{f})_i = (f, \phi_i)$.

Error Estimates

Theorem

Let $f \in \mathbb{L}^2(\Omega)$ and $\Psi \in H^2(\Omega)$, which implies $u \in H^2(\Omega)$, then there is a constant $c = \tilde{c}(\Omega, f, \Psi) > 0$ such that

$$|u - u^h|_0 \leq |u - u^h|_1 \leq ch \quad (20)$$

Interpolation estimate:

$$|v - v_I|_0 + h|v - v_I|_1 \leq ch^2 \text{ for all } v \in H^2(\Omega) \cap H^1_0(\Omega) \quad (21)$$

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Example I

We consider the free boundary problem for $\{u(x), x_0\}$ such that

$$\frac{d^2u}{dx^2} = 1 \; , 0 < x < x_0 < 1 \tag{22}$$

$$u = 0 \; , x_0 < x < 1 \tag{23}$$

with $u(0) = \alpha$, $0 < \alpha < 1/2$, $u(x_0) = \frac{du}{dx}(x_0) = 0$.
The corresponding linear complementary problem

$$-\frac{d^2u}{dx^2} + 1 \geq 0 \; , \; u \geq 0 \; , \; u(\frac{d^2u}{dx^2} + 1) = 0 \tag{24}$$

for almost all $x \in (0, 1)$, with $u(0) = \alpha$,$u(1) = 0$. Is used in a finite difference approximation to the problem.

Example II

Instead we try to discretize the constrained minimization problem

$$\arg\min_{v \in K} J[v] \; , \; J[v] = \frac{1}{2} \int_\Omega \nabla v \cdot \nabla v \; \mathrm{d}x\mathrm{d}y - \int_\Omega fv \; \mathrm{d}x\mathrm{d}y \qquad (25)$$

with using piecewise linear basis functions

$$\phi_i(x) = \begin{cases} (x - (i-1)h)/h \; , & x \in [(i-1)h, ih] \\ ((i+1)h - x)/h \; , & x \in [ih, (i+1)h] \end{cases}$$

with $i = 0, 1, \ldots, N$.

Example III

Then, for $v \in V^h$,

$$J[v] = \sum_{i=1}^{N} \int_{(i-1)h}^{ih} \frac{1}{2} \left(\frac{v_i - v_{i-1}}{h} \right)^2 + v_{i-1} + \frac{(x-(i-1)h)(v_i - v_{i-1})}{h} \mathrm{d}x \tag{26}$$

since on $[(i-1)h, ih]$

$$v = v_{i-1}\phi_{i-1} + v_i\phi_i \tag{27}$$

$$= \frac{1}{h}((ih - x)v_{i-1} + (x-(i-1)h)v_i) \tag{28}$$

$$= \frac{1}{h}((((i-1)h - x)v_{i-1} + (x-(i-1)h)v_i) + v_{i-1} \tag{29}$$

$$= \frac{1}{h}((x-(i-1)h)(v_i - v_{i-1})) + v_{i-1} \tag{30}$$

Example IV - error

Let l be the first mesh point at which the discrete solution vanishes.

Solution to continuous problem:

$$u(x) = \begin{cases} \frac{1}{2}(x - x_0)^2 \, , & 0 < x < x_0 \\ 0 \, , & x_0 < x < 1 \end{cases}$$

with $x_0 = \sqrt{2\alpha}$

Solution to discrete problem,

$$u_{i-1} - 2u_i + u_{i+1} = h^2 \, , 1 \leq \; i \leq l$$

$$u_0 = \alpha, \; u_l = 0, \; 0 < u_{l-1} \leq h^2$$

is

$$u_i = \frac{(i - l)(h^2 il - x_0^2)}{2l}$$

So the error

$$|u(ih) - u_i| = \frac{i(x_0 - lh)^2}{2l} = O(h^2) \tag{31}$$

where we used that $|x_0 - lh| < h$.

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Summary

1. Described the obstacle problem
2. Proved an a priori error estimate in the H^1 norm for the finite element discretization
3. Applied the discretization to a one-dimensional model problem

Bibliography

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