Complex variable methods in Hele-Shaw moving boundary problems

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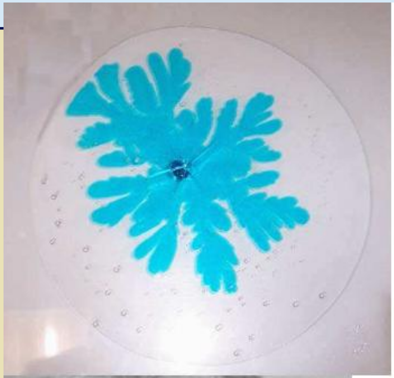
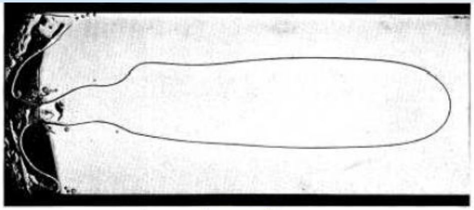
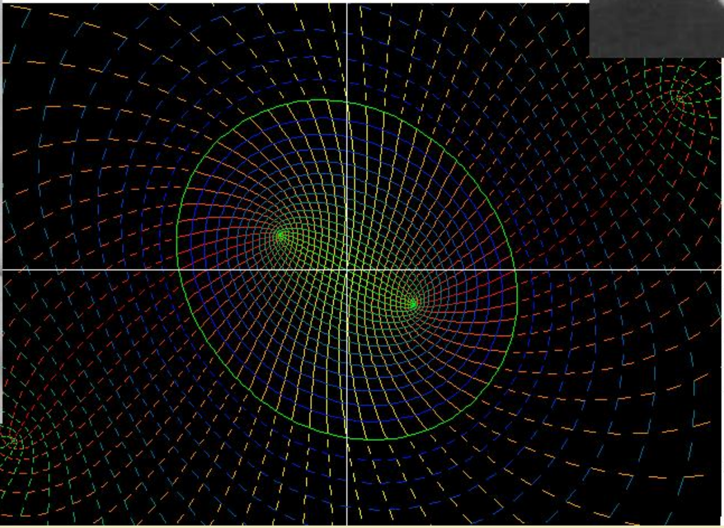
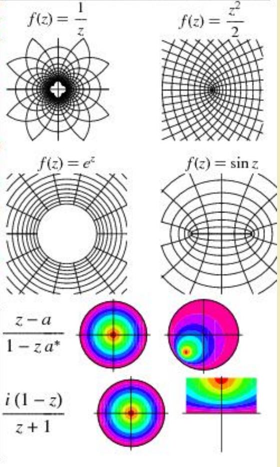




$$\mathrm{Re}\left(\frac{\partial F}{\partial \zeta}\overline{\frac{\partial F}{\partial t}}\right) = V$$

$$\mathrm{Re}\left(\zeta\frac{\partial f}{\partial \zeta}\overline{\frac{\partial f}{\partial t}}\right) = Q$$





Contents:

- Literature: article *Complex variable methods in Hele-Shaw moving boundary problems* by S. D. Howison
- Topics:
  - Moving boundary problems in 2D: Hele-Shaw
  - Complex analysis $\mathbb{R}^2 \cong \mathbb{C}$

1. Introduction

In two dimensions we consider the zero surface tension version of the Hele-Shaw problem (also called classical Hele-Shaw problem) in which the pressure $p = p(x, y, t)$ satisfies

$$\Delta p = 0$$

in the liquid region $\Omega(t)$, while on the free boundary $\partial\Omega(t)$ we have

$$p = 0.$$

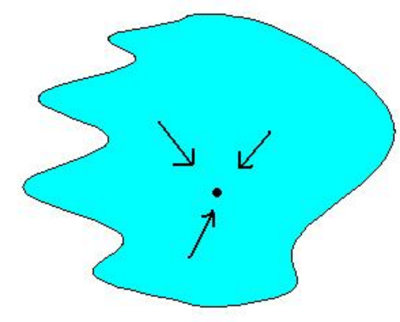
Boundary moves according to

$$V_n = -\frac{\partial p}{\partial n},$$

where V_n is the normal velocity.

- Complex variable theory provides an obvious starting point for an attack on the two-dimensional problem.
- Analytic functions:
  analyticity$\Leftrightarrow$ Cauchy Riemann equations $\Leftrightarrow$ differentiability $\Rightarrow$ real/imaginary part harmonic in x and y
- Conformal mapping methods
- Riemann Mapping Theorem

1) Flows generated by injection/suction from a single source/sink in a finite region



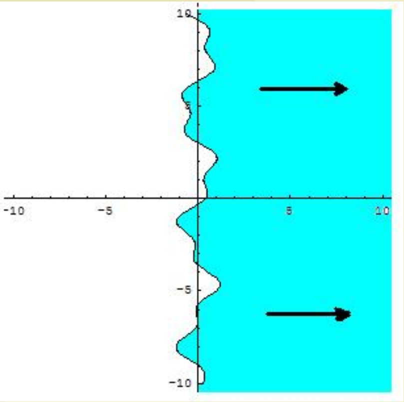
Extra assumption near origin:

$$p \sim Q \ln \sqrt{x^2 + y^2}, \quad x^2 + y^2 \to \infty$$

for some $Q \in \mathbb{R}$.

- $Q > 0$: suction
- $Q < 0$: injection

2) Flows with uniform suction at infinity:



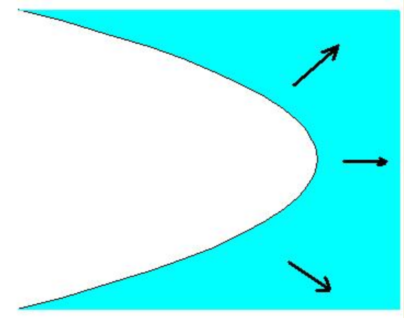
Extra assumption at infinity:

$$p \sim -Vx, \quad x^2 + y^2 \to \infty$$

for some $V \in \mathbb{R}$.

- $V > 0$: retreating fluid
- $V < 0$: expanding fluid

3) Flows in which $\partial\Omega(t)$ is approximately parabolic as $x \to \infty$:



Extra assumption at infinity:

$$p \sim -A\,\mathrm{Re}\sqrt{x+iy}, \quad x^2+y^2 \to \infty$$

for some $A \in \mathbb{R}$.

2. linear stability and regularisation

- traveling wave solution:

$$\Omega(t) = \{(x, y) : x > Vt\}$$

- small initial perturbation ($\delta << 1$)

$$\Omega(0) = \{(x, y) : x > \delta \sin(|n|y)\}$$

gives

$$\Omega(t) = \{(x, y) : x > Vt + \delta e^{\mu t} \sin(|n|y)\}$$

with

$$\mu = V|n|$$

(Saffman and Taylor 1958)

Variations for boundary conditions:

- Hele-Shaw flow with surface tension:

$$p = \gamma\kappa \quad \text{on } \partial\Omega(\mathrm{t})$$

  eigenvalues

$$\mu = V|n| - \gamma|n|^3$$

- Hele-Shaw flow with kinetic undercooling regularisation (as proposed by Romero (1982)):

$$p + \epsilon\frac{\partial p}{\partial n} = 0 \quad \text{on } \partial\Omega(\mathrm{t})$$

  eigenvalues

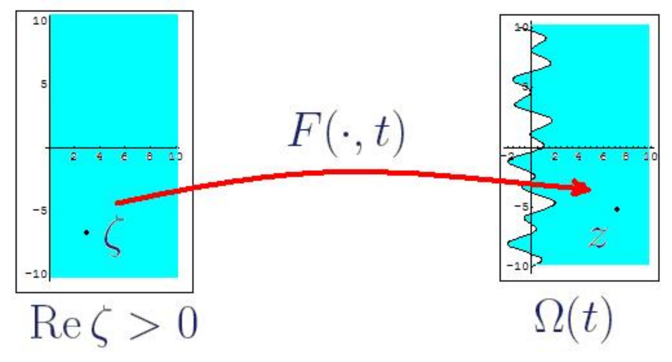
$$\mu = \frac{V|n|}{1 + \epsilon|n|}$$

3. Polubarinova-Galin equation for classical case

- Introduce complex potential w to be an analytic function with real part p:

$$w = p + iq.$$

- Introduce for infinite domain conformal mapping $z = F(\zeta, t)$ (see picture) from ζ-plane to z-plane



- In the ζ-plane the complex potential W is

$$W(\zeta,t) = w(F(\zeta,t),t) = -V\zeta.$$

We get

$$w = W \circ F^{-1}(\cdot,t) = -VF^{-1}(\cdot,t).$$

- We have the Polubarinova-Galin equation

$$\mathrm{Re}\left(\frac{\partial F}{\partial \zeta}\overline{\frac{\partial F}{\partial t}}\right) = V \quad \text{for } \xi = \mathrm{Re}\,\zeta = 0.$$

4. Traveling wave solutions for infinite domains

$$F(\zeta, t) = Ut + F_1(\zeta),$$

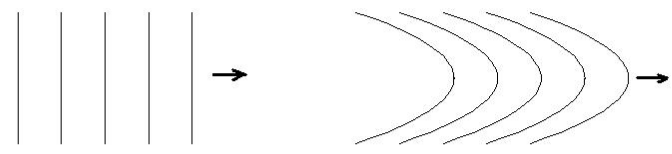
with $U \in \mathbb{R}$ (i.e. wave moves to the right). Polubarinova-Galin gives

$$\operatorname{Re} F_1' = V/U \quad \text{for } \operatorname{Re} \zeta = 0.$$

Two entire solutions:

$$F_1(\zeta) = \zeta \quad \text{with } U = V$$

$$F_1(\zeta) = \zeta^2 + C\zeta$$



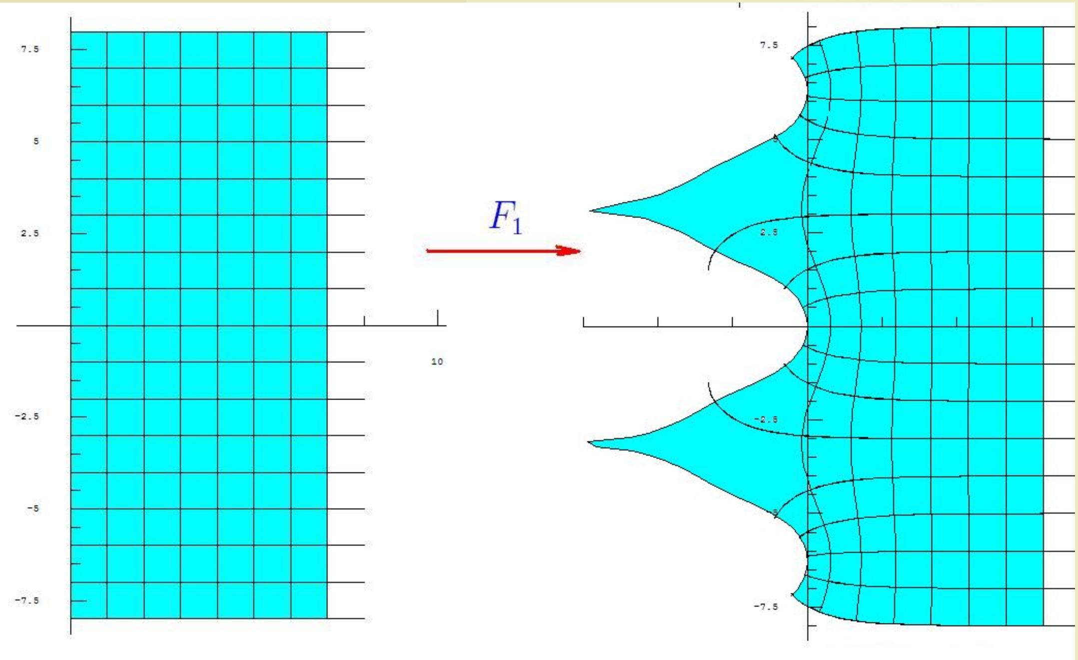
straight lines and parabolas

- Liouville's theorem and Schwarz reflection principle applied to $F_1 - \zeta$ and $F_1 - C\zeta$ show that these are the only solutions for which F_1 is analytic on $\{\mathrm{Re}\,\zeta > 0\} \cup \{\mathrm{Re}\,\zeta = 0\}$ (Howison et al. 1988).
- A famous example with poles at the imaginary axes at $\zeta = (2n+1)\pi i$ is the Saffman-Taylor finger (1958):

$$F_1'(\zeta) = \frac{V}{U} + 2\left(1 - \frac{V}{U}\right) \sum_{n=-\infty}^{\infty} \frac{1}{\zeta - (2n+1)\pi i}.$$

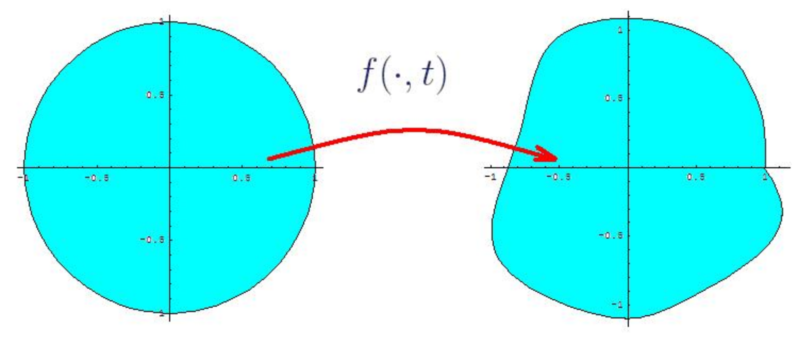
- We integrate this and obtain

$$F_1(\zeta) = \zeta + 2(1-\lambda)\ln\frac{1}{2}(1+e^{-\zeta})$$



5. polynomial conformal mappings for finite domain

- For the finite domain we consider conformal mappings $z = f(\zeta, t)$ from the unit disk $\{z : |z| < 1\}$ to $\Omega(t)$.



- The Polubarinova-Galin equation is

$$\mathrm{Re}\left(\zeta \frac{\partial f}{\partial \zeta}\overline{\frac{\partial f}{\partial t}}\right) = Q$$

- We consider polynomials (Polubarinova-Kochina 1945; Galin 1945; Richardson 1972)

$$f(\zeta, t) = \sum_{n=1}^{N} a_n(t)\zeta^n$$

- Substitution in the Polubarinova-Galin equation gives

$$\frac{d}{dt}\sum_{n=1}^{N} n|a_n|^2 = -2Q$$

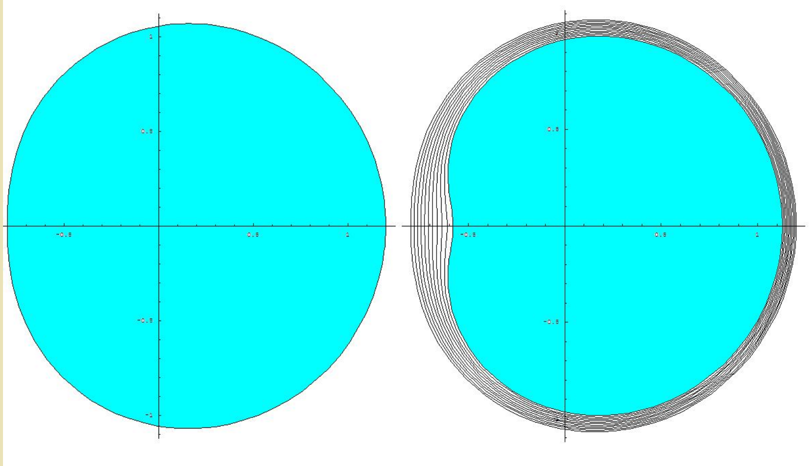
$$\sum_{n=1}^{N-k} n\overline{a_n}\frac{da_{k+n}}{dt} + (k+n)a_{k+n}\frac{d\overline{a_n}}{dt} = 0, \quad k \in \{1, 2, \ldots, N-1\}.$$

- special case: $N = 2$

$$f(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2.$$

- We get

$$\dot{a}_1 = Q\frac{a_1}{4a_2^2 - a_1^2} \quad \dot{a}_2 = -2Q\frac{a_2}{4a_2^2 - a_1^2}.$$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Thank you for your attention