Linear Multistep Methods

Lei Liu

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Technische Universiteit Eindhoven
University of Technology

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The Advection-Diffusion-Reaction equation

$$
\begin{aligned}
u_t + a u_x &= \varepsilon u_{xx} + \lambda u(1-u), \quad 0 < x < 1, \quad t > 0, \\
u_x(0,t) &= 0, \\
u(1,t) &= (1 + sin(\omega t))/2, \\
u(x,0) &= v(x).
\end{aligned}
$$

Parameters:

- a advection velocity
- ε diffusion coefficient
- λ source term coefficient

Definition

System of ODEs

$$w'(t) = F(t, w(t)), \quad t > 0, \quad w(0) = w_0.$$

The linear k-step method

$$\sum_{j=0}^{k} \alpha_j w_{n+j} = \tau \sum_{j=0}^{k} \beta_j F(t_{n+j}, w_{n+j}), \; n = 0, 1, \cdots .$$

- The method is explicit if $\beta_k = 0$,
- The method is implicit if $\beta_k \neq 0$.

Comparison with Runge-Kutta method

Advantages:

- In case of explicit methods only one F-evaluation is needed against s for the Runge-Kutta method.
- In case of implicit methods only one m dimensional system of nonlinear equations has to be solved against ms for the Runge-Kutta method.

Drawback:

- It needs k starting values for the first step.

Order condition

Inserting exact solution values in multistep formula yields

$$\sum_{j=0}^{k} \alpha_j w(t_{n+j}) = \tau \sum_{j=0}^{k} \beta_j F(t_{n+j}, w(t_{n+j})) + \tau \rho_{n+k-1}.$$

Taylor series expansion around $t = t_n$ yields

$$\tau \rho_{n+k-1} = C_0 w(t_n) + \tau C_1 w'(t_n) + \tau^2 C_2 w''(t_n) + \cdots ,$$

where

$$C_0 = \sum_{j=0}^{k} \alpha_j, \quad C_j = \frac{1}{i!}(\sum_{j=0}^{k} \alpha_j j^i - i \sum_{j=0}^{k} \beta_j j^{i-1}) \text{ for } i \geq 1.$$

Order condition

The method has order p iff it satisfies the *order conditions*

$$\sum_{j=0}^{k} \alpha_j = 0, \quad \sum_{j=0}^{k} \alpha_j j^i = i \sum_{j=0}^{k} \beta_j j^{i-1} \quad \textit{for } i = 1, 2, \cdots, p.$$

Classification and Examples

Adams-Bashforth methods

$$\alpha_k = 1, \quad \alpha_{k-1} = -1, \quad \alpha_j = 0 \ (0 \le j \le k-2)$$

- 2-step method

$$w_{n+2} - w_{n+1} = \frac{3}{2}\tau F_{n+1} - \frac{1}{2}\tau F_n$$

- 3-step method

$$w_{n+3} - w_{n+2} = \frac{23}{12}\tau F_{n+2} - \frac{16}{12}\tau F_{n+1} + \frac{5}{12}\tau F_n$$

Classification and Examples

Adams-Moulton methods

$$\alpha_k = 1, \quad \alpha_{k-1} = -1, \quad \alpha_j = 0 \ (0 \le j \le k-2)$$

- 2-step method

$$w_{n+2} - w_{n+1} = \frac{5}{12}\tau F_{n+2} + \frac{8}{12}\tau F_{n+1} - \frac{1}{12}\tau F_n$$

- 3-step method

$$w_{n+3} - w_{n+2} = \frac{9}{24}\tau F_{n+3} + \frac{19}{24}\tau F_{n+2} - \frac{5}{24}\tau F_{n+1} + \frac{1}{24}\tau F_n$$

Examples

BDF methods

$$\beta_k = 1, \quad \beta_j = 0 \ (0 \le j \le k-1)$$

- 2-step method

$$\frac{3}{2} w_{n+2} - 2 w_{n+1} + \frac{1}{2} w_n = \tau F_{n+2}$$

- 3-step method

$$\frac{11}{6} w_{n+3} - 3 w_{n+2} + \frac{3}{2} w_{n+1} - \frac{1}{3} w_n = \tau F_{n+3}$$

Convergence

$$\textit{Convergence} = \textit{Stability} + \textit{Consistency}.$$

Linear Recursions

The scalar linear recursion formula

$$\sum_{j=0}^{k} \gamma_j w_{n+j} = 0, \quad n = 0, 1, \cdots ,$$

with constant coefficients $\gamma_j \in \mathbb{C}$.
Characteristic polynomial

$$\pi(\zeta) = \sum_{j=0}^{k} \gamma_j \zeta^j.$$

with $\zeta_1, \zeta_2, \cdots , \zeta_k$ denoting zeros.

Root Condition

The characteristic polynomial is said to satisfy the *root condition* if

$$|\zeta_i| \leq 1 \ \text{ for all i, and } |\zeta_i| < 1 \quad \text{if } \zeta_i \text{ is not simple.}$$

The general solution of the recursion formula is

$$w_n = c_1 n^{\nu_1} \zeta_1^n + c_2 n^{\nu_2} \zeta_2^n + \cdots + c_k n^{\nu_k} \zeta_k^n, \quad n = 0, 1, \cdots .$$

Zero-Stability

For $\tau \to 0$ the linear multistep formula reduces to the linear recursion

$$\sum_{j=0}^{k} \alpha_j w_{n+j} = 0, \quad n = 0, 1, \cdots ,$$

with characteristic polynomial

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j .$$

The linear multistep method is said to be *zero-stable* if $\rho(\zeta)$ satisfies the root condition.

Zero-Stability

Remark: The maximal attainable order

- implicit k-step method $2k$
- explicit k-step method $2k - 1$

First Dahlquist Barrier for zero-stability

- implicit k-step method $2\lfloor (k+2)/2 \rfloor$
- explicit k-step method k

Stability Region

For the scalar stability test equation $w'(t) = \lambda w(t), \lambda \in \mathbb{C}$, we get the recursion

$$\sum_{j=0}^{k} (\alpha_j - z\beta_j) w_{n+j} = 0, \quad n = 0, 1, \cdots ,$$

where $z = \tau\lambda$. This recursion has the characteristic polynomial

$$\pi_z(\zeta) = \rho(\zeta) - z\sigma(\zeta) = \sum_{j=0}^{k} (\alpha_j - z\beta_j)\zeta^j.$$

Stability Region

The *stability region*

$$S = \{z \in \mathbb{C} | \text{ the recursion has bounded solution for any choice of starting values}\}.$$

Obviously,

$$z \in S \Leftrightarrow \pi_z \text{ satisfies the root condition.}$$

Stability Region

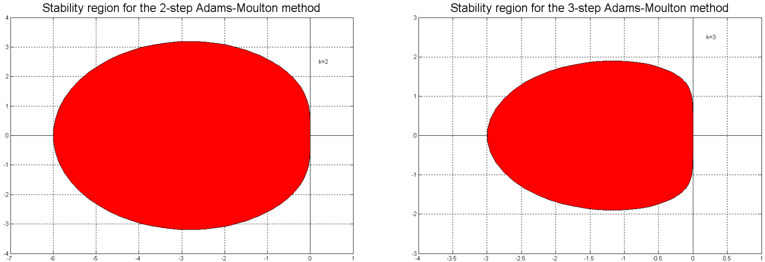

Figure: Stability region of 2-step and 3-step Adams-Moulton method

Stability Region

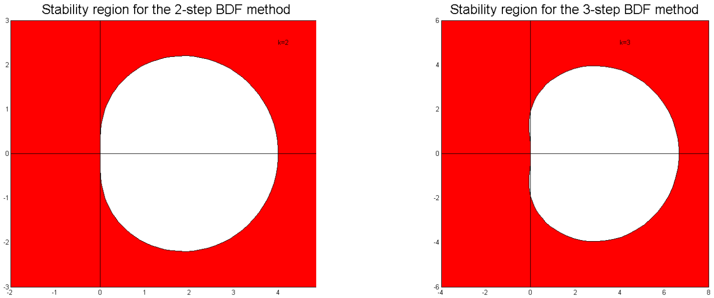

Figure: Stability region of 2-step and 3-step BDF method

A-Stability

Let $\overline{\mathbb{C}} = \mathbb{C} \bigcup \infty$, a linear multistep method is called *A-stable* if

$$S \supset \{z \in \overline{\mathbb{C}} : Re z \leq 0 \ or \ z = \infty\}.$$

Remark: *Second Dahlquist Barrier* states that the consistency order of an A-stable multistep method can be at most two.

Numerical Examples

$$
\begin{aligned}
u_t + au_x &= \varepsilon u_{xx} + \lambda u(1-u), \quad 0 < x < 1, \quad t > 0, \\
u_x(0,t) &= 0, \\
u(1,t) &= (1 + sin(\omega t))/2, \\
u(x,0) &= v(x).
\end{aligned}
$$

Parameters:

$$a = -1, \ \varepsilon = 0.01, \ \omega = 10, \ \lambda = 1$$

Initial condition:

$$u(x,0) = 0.5$$

Firstly we choose the time interval $[0, 1]$, and the number of grid points 101, so time step $\tau = 1/100$.

First movie is for AM method, second movie is for BDF method.

Next we choose the time interval for AM method $[0, 2]$ and for BDF method $[0, 5]$, and the number of grid points 101, so time steps are $1/50$ and $1/20$ respectively.

First movie is for AM method, second movie is for BDF method.

Last we choose the time interval for AM method $[0,3]$ and for BDF method $[0,10]$, and the number of grid points 101. so time steps are $3/100$ and $10/100$ respectively.

First movie is for AM method, second movie is for BDF method.

Richardson extrapolation

By Richardson extrapolation we estimate the convergence order p and the constant c_p such that

$$u = u_k + c_p h_k^p + O(h_k^{p+1})$$

for 2-step Adams-Moulton method and get better approximations. We choose the value at the point $(0.5, 1)$ calculated under the time steps $1/50, 1/100, 1/200, 1/400, 1/800$.

Richardson Extrapolation

The result

τ	u_k	p	c_p	$u_{richardson}$
1/50	0.294691915			
1/100	0.295320710			
1/200	0.295398920	3.007	97.985	0.29541002918
1/400	0.295408659	3.005	97.215	0.29541004452
1/800	0.295409873	3.004	96.584	0.29541004589

We get the order of 2-step AM method is 3, and by the same way we can get the order of 2-step BDF method is 2.

Conclusion

From the numerical experiments we can conclude

- AM method has third convergence order, but the numerical solution isn't stable for large time steps.
- BDF method has second convergence order, and it has favorable stability property.

These are according with the analytical results.

Thank you for your attention!