

# Linear Multistep Methods

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## The Advection-Diffusion-Reaction equation

$$u_t + au_x = \varepsilon u_{xx} + \lambda u(1 - u), \quad 0 < x < 1, \quad t > 0,$$

$$u_x(0, t) = 0,$$

$$u(1, t) = (1 + \sin(\omega t))/2,$$

$$u(x, 0) = v(x).$$

Parameters:

- $a$  advection velocity
- $\varepsilon$  diffusion coefficient
- $\lambda$  source term coefficient

## Definition

System of ODEs

$$w'(t) = F(t, w(t)), \quad t > 0, \quad w(0) = w_0.$$

The linear  $k$ -step method

$$\sum_{j=0}^k \alpha_j w_{n+j} = \tau \sum_{j=0}^k \beta_j F(t_{n+j}, w_{n+j}), \quad n = 0, 1, \dots$$

- The method is explicit if  $\beta_k = 0$ ,
- The method is implicit if  $\beta_k \neq 0$ .

## Comparison with Runge-Kutta method

Advantages:

- In case of explicit methods only one  $F$ -evaluation is needed against  $s$  for the Runge-Kutta method.
- In case of implicit methods only one  $m$  dimensional system of nonlinear equations has to be solved against  $ms$  for the Runge-Kutta method.

Drawback:

- It needs  $k$  starting values for the first step.

## Order condition

Inserting exact solution values in multistep formula yields

$$\sum_{j=0}^k \alpha_j w(t_{n+j}) = \tau \sum_{j=0}^k \beta_j F(t_{n+j}, w(t_{n+j})) + \tau \rho_{n+k-1}.$$

Taylor series expansion around  $t = t_n$  yields

$$\tau \rho_{n+k-1} = C_0 w(t_n) + \tau C_1 w'(t_n) + \tau^2 C_2 w''(t_n) + \dots,$$

where

$$C_0 = \sum_{j=0}^k \alpha_j, \quad C_j = \frac{1}{j!} \left( \sum_{j=0}^k \alpha_j j^j - i \sum_{j=0}^k \beta_j j^{i-1} \right) \text{ for } i \geq 1.$$

## Order condition

The method has order  $p$  iff it satisfies the *order conditions*

$$\sum_{j=0}^k \alpha_j = 0, \quad \sum_{j=0}^k \alpha_j j^i = i \sum_{j=0}^k \beta_j j^{i-1} \quad \text{for } i = 1, 2, \dots, p.$$

# Classification and Examples

## Adams-Bashforth methods

$$\alpha_k = 1, \quad \alpha_{k-1} = -1, \quad \alpha_j = 0 \quad (0 \leq j \leq k-2)$$

- 2-step method

$$w_{n+2} - w_{n+1} = \frac{3}{2}\tau F_{n+1} - \frac{1}{2}\tau F_n$$

- 3-step method

$$w_{n+3} - w_{n+2} = \frac{23}{12}\tau F_{n+2} - \frac{16}{12}\tau F_{n+1} + \frac{5}{12}\tau F_n$$



# Classification and Examples

## Adams-Moulton methods

$$\alpha_k = 1, \quad \alpha_{k-1} = -1, \quad \alpha_j = 0 \quad (0 \leq j \leq k-2)$$

- 2-step method

$$w_{n+2} - w_{n+1} = \frac{5}{12}\tau F_{n+2} + \frac{8}{12}\tau F_{n+1} - \frac{1}{12}\tau F_n$$

- 3-step method

$$w_{n+3} - w_{n+2} = \frac{9}{24}\tau F_{n+3} + \frac{19}{24}\tau F_{n+2} - \frac{5}{24}\tau F_{n+1} + \frac{1}{24}\tau F_n$$

## Examples

### BDF methods

$$\beta_k = 1, \quad \beta_j = 0 \quad (0 \leq j \leq k - 1)$$

- 2-step method

$$\frac{3}{2}w_{n+2} - 2w_{n+1} + \frac{1}{2}w_n = \tau F_{n+2}$$

- 3-step method

$$\frac{11}{6}w_{n+3} - 3w_{n+2} + \frac{3}{2}w_{n+1} - \frac{1}{3}w_n = \tau F_{n+3}$$

# Convergence

*Convergence = Stability + Consistency.*

# Linear Recursions

The scalar linear recursion formula

$$\sum_{j=0}^k \gamma_j w_{n+j} = 0, \quad n = 0, 1, \dots,$$

with constant coefficients  $\gamma_j \in \mathbb{C}$ .

Characteristic polynomial

$$\pi(\zeta) = \sum_{j=0}^k \gamma_j \zeta^j.$$

with  $\zeta_1, \zeta_2, \dots, \zeta_k$  denoting zeros.

## Root Condition

The characteristic polynomial is said to satisfy the *root condition* if

$$|\zeta_i| \leq 1 \text{ for all } i, \text{ and } |\zeta_i| < 1 \text{ if } \zeta_i \text{ is not simple.}$$

The general solution of the recursion formula is

$$w_n = c_1 n^{\nu_1} \zeta_1^n + c_2 n^{\nu_2} \zeta_2^n + \cdots + c_k n^{\nu_k} \zeta_k^n, \quad n = 0, 1, \dots .$$

## Zero-Stability

For  $\tau \rightarrow 0$  the linear multistep formula reduces to the linear recursion

$$\sum_{j=0}^k \alpha_j w_{n+j} = 0, \quad n = 0, 1, \dots,$$

with characteristic polynomial

$$\rho(\zeta) = \sum_{j=0}^k \alpha_j \zeta^j.$$

The linear multistep method is said to be *zero-stable* if  $\rho(\zeta)$  satisfies the root condition.

## Zero-Stability

Remark: The maximal attainable order

- implicit  $k$ -step method  $2k$
- explicit  $k$ -step method  $2k - 1$

*First Dahlquist Barrier* for zero-stability

- implicit  $k$ -step method  $2\lfloor(k + 2)/2\rfloor$
- explicit  $k$ -step method  $k$

## Stability Region

For the scalar stability test equation  $w'(t) = \lambda w(t)$ ,  $\lambda \in \mathbb{C}$ , we get the recursion

$$\sum_{j=0}^k (\alpha_j - z\beta_j) w_{n+j} = 0, \quad n = 0, 1, \dots,$$

where  $z = \tau\lambda$ . This recursion has the characteristic polynomial

$$\pi_z(\zeta) = \rho(\zeta) - z\sigma(\zeta) = \sum_{j=0}^k (\alpha_j - z\beta_j)\zeta^j.$$



# Stability Region

The *stability region*

$S = \{z \in \mathbb{C} \mid \text{the recursion has bounded solution for any choice of starting values}\}.$

Obviously,

$z \in S \Leftrightarrow \pi_z$  satisfies the root condition.

# Stability Region

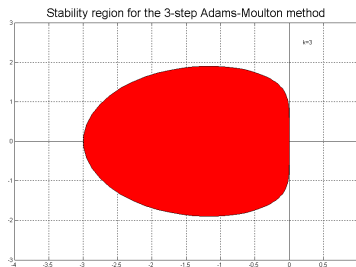
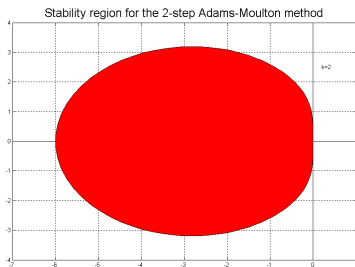


Figure: Stability region of 2-step and 3-step Adams-Moulton method

# Stability Region

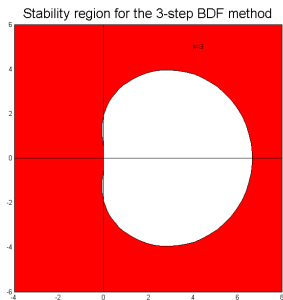
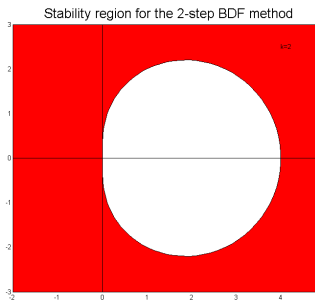


Figure: Stability region of 2-step and 3-step BDF method

# A-Stability

Let  $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ , a linear multistep method is called *A-stable* if

$$S \supset \{z \in \bar{\mathbb{C}} : \operatorname{Re} z \leq 0 \text{ or } z = \infty\}.$$

Remark: *Second Dahlquist Barrier* states that the consistency order of an A-stable multistep method can be at most two.

## Numerical Examples

$$\begin{aligned}u_t + au_x &= \varepsilon u_{xx} + \lambda u(1 - u), & 0 < x < 1, & t > 0, \\u_x(0, t) &= 0, \\u(1, t) &= (1 + \sin(\omega t))/2, \\u(x, 0) &= v(x).\end{aligned}$$

Parameters:

$$a = -1, \varepsilon = 0.01, \omega = 10, \lambda = 1$$

Initial condition:

$$u(x, 0) = 0.5$$

Firstly we choose the time interval  $[0, 1]$ , and the number of grid points 101, so time step  $\tau = 1/100$ .

First movie is for AM method, second movie is for BDF method.

Next we choose the time interval for AM method  $[0, 2]$  and for BDF method  $[0, 5]$ , and the number of grid points 101, so time steps are  $1/50$  and  $1/20$  respectively.

First movie is for AM method, second movie is for BDF method.

Last we choose the time interval for AM method  $[0, 3]$  and for BDF method  $[0, 10]$ , and the number of grid points 101. so time steps are  $3/100$  and  $10/100$  respectively.

First movie is for AM method, second movie is for BDF method.



## Richardson extrapolation

By Richardson extrapolation we estimate the convergence order  $p$  and the constant  $c_p$  such that

$$u = u_k + c_p h_k^p + O(h_k^{p+1})$$

for 2-step Adams-Moulton method and get better approximations. We choose the value at the point  $(0.5, 1)$  calculated under the time steps  $1/50, 1/100, 1/200, 1/400, 1/800$ .

## Richardson Extrapolation

### The result

$\tau$	$u_k$	$p$	$C_p$	$u_{richardson}$
1/50	0.294691915			
1/100	0.295320710			
1/200	0.295398920	3.007	97.985	0.29541002918
1/400	0.295408659	3.005	97.215	0.29541004452
1/800	0.295409873	3.004	96.584	0.29541004589

We get the order of 2-step AM method is 3, and by the same way we can get the order of 2-step BDF method is 2.

## Conclusion

From the numerical experiments we can conclude

- AM method has third convergence order, but the numerical solution isn't stable for large time steps.
- BDF method has second convergence order, and it has favorable stability property.

These are according with the analytical results.

**Thank you for your attention!**