Casa seminar

Rosenbrock (AMF) methods

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Contents

Introduction

$$u_t + (au)_x = (du_x)_x + f(u),$$

Runge-Kutta methods

Runge-Kutta methods:

We have $\dfrac{dw}{dt} = F(w)$.

$$\begin{cases} w_{n+1} = w_n + \sum_{i=1}^{s} b_i k_i \\ k_i = \tau F\left(w_n + \sum_{j=1}^{s} \alpha_{ij} k_j\right) \qquad\qquad i = 1, \dots, s \end{cases}$$

Explicit if $\alpha_{ij} = 0$ for $j \geq i$.

Otherwise implicit.

Diagonal Implicit Runge-Kutta

This method is

$$\begin{cases} w_{n+1} = w_n + \displaystyle\sum_{i=1}^{s} b_i k_i \\ k_i = \tau F\left(w_n + \displaystyle\sum_{j=1}^{i} \alpha_{ij} k_j\right) \qquad i = 1, \ldots, s \end{cases}$$

Newton iteration to calculate k_i

Rosenbrock Methods

Rosenbrock methods are Runge-Kutta type methods.

$$\begin{cases} w_{n+1} = w_n + \sum_{i=1}^{s} b_i k_i \\ k_i = \tau F\left(w_n + \sum_{j=1}^{i-1} \alpha_{ij} k_j\right) + \tau J \sum_{j=1}^{i} \gamma_{ij} k_j \qquad i = 1, \ldots, s \end{cases}$$

With $J = F'(w_n)$.

Jacobian matrix in the integration formula.

Rosenbrock Methods (II)

A-stability or L-stability

order p	order conditions
1	$b_1 + b_2 + b_3 = 1$
2	$b_2 d_2 + b_3 d_3 + b_4 d_4 = \frac{1}{2} - \gamma$
3	$b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{3}$ $b_3 \beta_{32} d_2 + b_4 (\beta_{42} d_2 + \beta_{43} d_3) = \frac{1}{6} - \gamma + \gamma^2$

where

$$\beta_{ij} = \alpha_{ij} + \gamma_{ij}, \quad c_i = \sum_{j=1}^{i-1} \alpha_{ij}, \quad d_i = \sum_{j=1}^{i-1} \beta_{ij}.$$

Rosenbrock Methods (III)

The 1-stage method

$$w_{n+1} = w_n + k_1, \quad k_1 = \tau F(w_n) + \gamma \tau J k_1.$$

Stability function

$$R(z) = \frac{1 + (1-\gamma)z}{1 - \gamma z}$$

Order 2 if $\gamma = \frac{1}{2}$.

A-stability if $\gamma \geq \frac{1}{2}$ and L-stability if $\gamma = 1$.

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Rosenbrock Methods (IV)

The 2-stage method

We have

$$\begin{aligned} w_{n+1} &= w_n + b_1 k_1 + b_2 k_2, \\ k_1 &= \tau F(w_n) + \gamma \tau J k_1 \\ k_2 &= \tau F(w_n + \alpha_{21} k_1) + \gamma_{21} \tau J k_1 + \gamma \tau J k_2 \end{aligned}$$

with

$$b_1 = 1 - b_2, \quad \alpha_{21} = \frac{1}{2b_2}, \quad \gamma_{21} = -\frac{\gamma}{b_2}$$

Stability function

$$R(z) = \frac{1 + (1-2\gamma)z + (\frac{1}{2} - 2\gamma + \gamma^2)z^2}{(1-\gamma z)^2}$$

A-stability if $\gamma \geq \frac{1}{4}$ and L-stability if $\gamma = 1 \pm \frac{1}{2}\sqrt{2}$.

Rosenbrock AMF methods

Splitting based on special approximations of the Jacobian matrix.

$$w'(t) = F(w(t)) = F_1(w(t)) + \cdots + F_s(w(t))$$

One-stage method

$$w_{n+1} = w_n + k_i, \qquad k_i = \tau(I - \gamma\tau J)^{-1} F(w_n)$$

$$w_{n+1} = w_n + \tau B^{-1} F(w_n), \quad B = I - \gamma\tau J,$$

$J = F'(w_n)$.

Replace $B = I - \gamma\tau J$ by

$$\tilde{B} = (I - \gamma\tau J_1)(I - \gamma\tau J_2)\cdots(I - \gamma\tau J_s)$$

Model problem

$$\begin{aligned}
u_t + au_x &= \varepsilon u_{xx} + \lambda u(1-u), & 0 < x < 1,\, t > 0\\
u_x(0,t) &= 0, & t > 0\\
u(1,t) &= (1+\sin(wt)/2, & t > 0\\
u(x,0) &= v(x), & 0 < x < 1.
\end{aligned}$$

Write problem as ordinary differential equation.

Model problem

The ODE becomes

$$\overline{u_t} = \frac{\varepsilon}{h^2}\begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}\overline{u} - \frac{a}{2h}\begin{pmatrix} 0 & 0 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}\overline{u}$$

$$+ \left(\frac{\varepsilon}{h^2} - \frac{a}{2h}\right)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\left(\frac{1+\sin(wt)}{2}\right) + \lambda(1-u_j)u_j$$

$$= (A_d - A_a)\overline{u} + b + f = F(\overline{u})$$

Now apply Rosenbrock (AMF) methods.

Numerical results

Rosenbrock methods

The 1-stage method

$$w_{n+1} = w_n + k_1, \quad k_1 = \tau F(w_n) + \gamma\tau J k_1.$$

J is the Jacobian matrix $F'(w_n)$.
$a = -1, \varepsilon = 0.01, \lambda = 1,$ frequency $= 10, N_x = 100, N_t = 100$.

Unstable $\gamma = 0.1$

Unstable $\gamma = 0.2$.

Numerical results

Stable $\gamma = 0.4$

Stable $\gamma = 0.6$.

Stable $\gamma = 1.2$

Stable $\gamma = 2.0$.

Numerical results

$a = -1, \varepsilon = 1, \lambda = 1,$ frequency $= 10, N_x = 100, N_t = 100$.

Unstable $\gamma = 0.4$

Stable $\gamma = 0.5$.

The 2-stage method

$$\begin{aligned} w_{n+1} &= w_n + b_1 k_1 + b_2 k_2, \\ k_1 &= \tau F(w_n) + \gamma \tau J k_1 \\ k_2 &= \tau F(w_n + \alpha_{21} k_1) + \gamma_{21} \tau J k_1 + \gamma \tau J k_2 \end{aligned}$$

with

$$b_1 = 1 - b_2, \quad \alpha_{21} = \frac{1}{2b_2}, \quad \gamma_{21} = -\frac{\gamma}{b_2}$$

Numerical results

$a = -1, \varepsilon = 0.01, \lambda = 1,$ frequency $= 10, b_2 = 0.4, N_x = 100, N_t = 100$.

Unstable $\gamma = 0.1$

Stable $\gamma = 0.2$.

Numerical results

Stable $\gamma = 0.3$

Stable $\gamma = 0.5$.

Stable $\gamma = 1.0$

Stable $\gamma = 1.5$.

Numerical results

$a = -1, \varepsilon = 1, \lambda = 1,$ frequency $= 10, b_2 = 0.4, N_x = 100, N_t = 100$.

Unstable $\gamma = 0.2$

Stable $\gamma = 0.3$.

Rosenbrock AMF methods

One-stage method

$$w_{n+1} = w_n + \tau B^{-1} F(w_n), \quad B = I - \gamma\tau J,$$

$J = F'(w_n)$.

Replace $B = I - \gamma\tau J$ by

$$\tilde{B} = (I - \gamma\tau J_1)(I - \gamma\tau J_2)\cdots(I - \gamma\tau J_s)$$

Approximation is

$$\tilde{B} = (I - \gamma\tau A)(I - \gamma\tau f') = B + (\gamma\tau)^2 A f' = B + \mathcal{O}(\tau^2)$$

Numerical results

$a = -1, \varepsilon = 0.01, \lambda = 1,$ frequency $= 10, b_2 = 0.4, N_x = 100, N_t = 100$.

Unstable $\gamma = 0.1$

Unstable $\gamma = 0.2$.

Numerical results

Stable $\gamma = 0.3$

Stable $\gamma = 0.4$.

Stable $\gamma = 0.6$

Stable $\gamma = 1.2$.

Numerical results

$a = -1, \varepsilon = 1, \lambda = 1,$ frequency $= 10, b_2 = 0.4, N_x = 100, N_t = 100$.

$\varepsilon = 1$, Unstable $\gamma = 0.4$

Stable $\gamma = 0.5$.

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Conclusion

Rosenbrock 1-stage method

A-Stability when $\gamma \geq \frac{1}{2}$.

Rosenbrock 2-stage method

A-Stability when $\gamma \geq \frac{1}{4}$.

Rosenbrock AMF method

A-Stability when $\gamma \geq \frac{1}{2}$.

Questions

Questions?

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