LINEAR DISPERSIVE WAVES

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December 16, 2009

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Outline

Dispersion Relation

Ansatz:

$$\varphi(x,t) = Ae^{(i\kappa x - i\omega t)},$$

κ, ω, A are constants.
For κ, ω,

$$G(\omega,\kappa) = 0$$

is the *dispersive relation.*

$$\omega = \mathbf{W}(\kappa),$$

are called modes.

$$\mathbf{Re}(\varphi) = \mid A \mid \cos(\kappa x - \omega t + \eta),\ \eta = arg(A),$$

$\theta = \kappa x - \omega t$ determines $\mathbf{Re}(\varphi)$.

Dispersion Relation

Hence,

$$\theta_x = \kappa,\ -\theta_t = \omega,\ \Rightarrow \lambda = \frac{2\pi}{\kappa}, \tau = \frac{2\pi}{\omega}.$$

The phase velocity:

$$c = \frac{\omega}{\kappa}, \qquad \frac{\omega}{\kappa}$$

normal to κ.
We need

$$\text{determinant}\left| \frac{\partial^2 \mathbf{W}}{\partial \kappa_i \partial \kappa_j} \right| \neq 0,$$

for $W(\kappa)$ real. In 1D

$$W''(\kappa) \neq 0.$$

Dispersion Relation

Examples:

Klein-Gordon (quantum theory):

$$\varphi_{tt} - \alpha^2 \nabla^2 \varphi + \beta^2 \varphi = 0, \; \omega = \pm\sqrt{\alpha^2 \kappa^2 + \beta^2};$$

Korteweg-de Vries equation (Long water waves):

$$\varphi_t + \alpha \varphi_x + \beta \varphi_{xxx} = 0, \; \omega = \alpha\kappa - \beta\kappa^3;$$

Boussinesq equation (Longitudinal waves for elasticity):

$$\varphi_{tt} - \alpha^2 \nabla^2 \varphi = \beta^2 \nabla^2 \varphi_{tt}, \omega = \pm \frac{\alpha\kappa}{\sqrt{1 + \beta^2 \kappa^2}};$$

Dispersive Relations

The Beam equation

$$\varphi_{tt} + \gamma^2 \varphi_{xxxx} = 0,\ \omega^2 - \gamma^2 \kappa^4 = 0,$$

with modes $\omega = \pm\gamma\kappa^2$.
Generally,

$$p\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = 0,$$

and P a polynomial. Hence,

$$\frac{\partial}{\partial x_j} \to i\kappa_j, \quad \frac{\partial}{\partial t} \to -i\omega.$$

and

$$P(-i\omega, i\kappa_1, i\kappa_2, i\kappa_3) = 0,\ \Rightarrow \frac{\partial}{\partial t} \leftrightarrow -i\omega, \frac{\partial}{\partial x_j} \leftrightarrow i\kappa_j.$$

Definition of Dispersive Waves

A linear dispersive wave satisfies

$$\varphi(x,t) = Ae^{i\kappa x - i\omega t}, \quad \omega = W(\kappa),$$

and

$$\text{determinant} \left| \frac{\partial^2 W}{\partial \kappa_i \partial \kappa_j} \right| \neq 0.$$

Hence when oscillations in space are coupled with oscillations in time through a dispersion relation, we expect typical effects of Dispersive Waves.

Outline

The Beam Equation

Consider

$$\varphi_{tt} + \gamma^2 \varphi_{xxxx} = 0,$$

recall

$$\omega^2 - \gamma^2 \kappa^4 = 0, \Rightarrow \omega = \pm\gamma\kappa^2,$$

From

$$\varphi(x,t) = \int_{-\infty}^{\infty} F(\kappa) e^{(i\kappa x - iW(\kappa)t)} \mathrm{d}\kappa,$$

$F(\kappa)$ depends on φ, φ_t.

General Solution by Fourier Integrals

Solution:

$$\varphi(x,t) = \int_{-\infty}^{\infty} F_1(\kappa) e^{(i\kappa x - iW(\kappa)t)} + \int_{-\infty}^{\infty} F_2(\kappa) e^{(i\kappa x + iW(\kappa)t)} \mathrm{d}\kappa$$

with

$$\varphi = \varphi_0(x),\ \varphi_t = \varphi_1(x).$$

we get

$$\varphi_0(x) = \int_{-\infty}^{\infty} (F_1(\kappa) + F_2(\kappa)) e^{i\kappa x} \mathrm{d}\kappa,$$

and

$$\varphi_1(x) = -i \int_{-\infty}^{\infty} W(\kappa)(F_1(\kappa) - F_2(\kappa)) e^{i\kappa x} \mathrm{d}\kappa.$$

Solution by Fourier Integrals

Inverse formula:

$$F_1(\kappa) + F_2(\kappa) = \Phi_0(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_0(x) e^{-i\kappa x} \mathrm{d}x.$$

$$-iW(\kappa)\left[F_1(\kappa) - F_2(\kappa)\right] = \Phi_1(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_1(x) e^{-i\kappa x} \mathrm{d}x.$$

which gives

$$F_1(\kappa) = \frac{1}{2}\left[\Phi_0 + \frac{i\Phi_1(\kappa)}{W(\kappa)}\right], F_2(\kappa) = \frac{1}{2}\left[\Phi_0 - \frac{i\Phi_1(\kappa)}{W(\kappa)}\}\right].$$

Since $\varphi_0(x),\ \varphi_1(x)$ are real $\Phi_0(-\kappa) = \Phi_0^*(\kappa),\ \Phi_1(-\kappa) = \Phi_1^*(\kappa)$,

$$F_1(-\kappa) = F_1^*(\kappa),\ F_2(-\kappa) = F_2^*(\kappa),\ \text{if } W(-\kappa) = -W(\kappa),$$

$$F_1(-\kappa) = F_2^*(\kappa),\ F_2(-\kappa) = F_1^*(\kappa),\ \text{if } W(-\kappa) = W(\kappa).$$

Asymptotic Behaviour

From

$$\varphi(x,t) = \int_{-\infty}^{\infty} F(\kappa) e^{i\kappa x - iW(\kappa)t} \mathrm{d}\kappa,$$

x/t fixed , $t \rightarrow \infty$,

$$\varphi(x,t) = \int_{-\infty}^{\infty} F(\kappa) e^{-i\chi t} \mathrm{d}\kappa, \ \chi = W(\kappa) - \kappa \frac{x}{t}.$$

around $\kappa = k$ and

$$\chi^{'}(\kappa) = W^{'}(\kappa) - \frac{x}{t} = 0.$$

By Taylor expansion of $\chi(\kappa)$ and $F(\kappa)$,

$$F(\kappa) \simeq \ F(k), \ \chi(\kappa) = \chi(k) + (\kappa - k)^2 \chi^{''}(\kappa), \chi^{''}(\kappa) \neq 0.$$

Asymptotic Behaviour

We obtain

$$\varphi \sim F(k)e^{-i\chi(\kappa)t}\int_{-\infty}^{\infty}\exp\left(-\frac{i}{2}(\kappa-k)^2\chi''(\kappa)t\right)\mathrm{d}\kappa,$$

from

$$\int_{-\infty}^{\infty}e^{-\alpha z^2}\mathrm{d}z=\sqrt{\frac{\pi}{\alpha}},$$

(path through $\pm\pi/4$),

$$\varphi \sim F(k)\sqrt{\frac{2\pi}{t\mid\chi''(k)\mid}}\exp -i\chi(k)t-i\frac{\pi}{4},$$

for Beam equation.

Asymptotic Behaviour

From above,

$$k(x,t): \ W(k) = \frac{x}{t}, \ k > 0, \frac{x}{t} > 0, \ W(k) = -W(k),,$$

If

$$W^{'}(k) = const, \ W^{''}(\kappa) = 0,$$

we need $\chi^{'''}(\kappa) \neq 0$, and

$$\varphi \sim F(k)e^{-i\chi(k)t}\int_{-\infty}^{\infty}\exp(-\frac{i}{6}\chi^{'''}(\kappa)t(\kappa-k)^3)\mathrm{d}\kappa.$$

So $W^{''}(\kappa) = 0, \ \kappa = const$, equation is undefined on $k \sim k(x/t) \ \rightarrow x/t = W^{'}(k)$.

Outline

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Kinematic Derivation of Group Velocity

Generally, for (x, t),

$$\theta(x,t) = xk(x,t) - t\omega(x,t),$$

$k = \theta_x,\ \omega = -\theta_t,$

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial t} = 0,$$

$k(x,t)$ (density) and ω (flux). For $\omega = W(k)$,

$$\frac{\partial k}{\partial t} + C(k)\frac{\partial k}{\partial x} = 0,\ C(k) = W^{'}(k),$$

$C(k)$, for some k, if $k = f(x),\ t = 0$,

$$k = f(\xi),\ x = \xi + C(\xi)t,\ C(\xi) = C(f(\xi)).$$

gives k from (1), and $x = C(k)t$, $C(k)$ is the group velocity.

Kinematics Derivation of Group Velocity

Hence, for a k_0, $x = W'(k_0)t$,

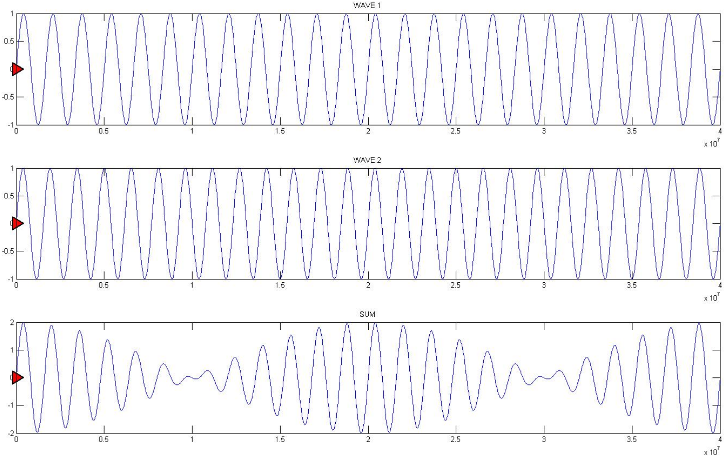
$$W'(k) = \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

is the group velocity.
For $\theta(x,t) = \theta_0$,

$$\theta_x \frac{\mathrm{d}x}{\mathrm{d}t} + \theta_t = 0 \;\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\theta_t}{\theta_x} = \frac{\omega}{k}$$

defines the phase velocity.

Example



Kinematics Derivation of Group Velocity

Examples:

1) Beam Equation gives:

$$W(k) = \gamma k^2 \Rightarrow W^{'}(k) = 2\gamma k = \frac{x}{t},$$

and

$$k = \frac{x}{2\gamma t}, \quad \omega = \frac{x^2}{4\gamma t^2}, \quad \theta = \frac{x^2}{4\gamma t.}$$

$$c_g = 2\gamma k, \quad c_p = \gamma k \;\Rightarrow c_g > c_p.$$

Group and phase lines:

$$\frac{x}{2\gamma t} = const, \quad \frac{x^2}{4\gamma t} = const.$$

Kinematics Derivation of Group Velocity

2) Deep water waves:

$$W(k) = \sqrt{gk} \Rightarrow W'(k) = \sqrt{\frac{g}{4k}},$$

$$k = \frac{gt^2}{4x^2}, \quad \omega = \frac{gt^2}{2x}, \ \theta = -\frac{gt^2}{4x}$$

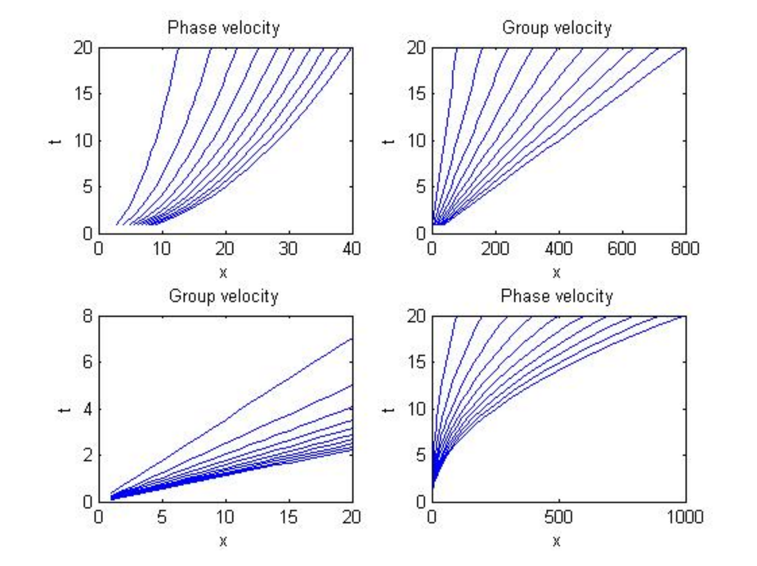
implies $c_g < c_p$. where

$$c_g = \frac{1}{2}\sqrt{\frac{g}{k}}, \quad c_p = \sqrt{\frac{g}{k}}.$$

Thus,

$$\frac{gt^2}{4x} = const, \quad \frac{x}{2\gamma t} = const.$$

Output for Group and Phase Velocities



Extension to Higher Dimensions

General solution:

$$\varphi = \int_{\mathbf{R}^n} F(\kappa) e^{i\kappa.\mathbf{x} - i\mathbf{W}(\kappa)t} \mathrm{d}\kappa,$$

$$\sim F(\kappa) \left(\frac{2\pi}{t}\right)^{\frac{n}{2}} \left(det \left|\frac{\partial W}{\partial k_i \partial k_j}\right|\right)^{-\frac{1}{2}} e^{ikx - i\mathbf{W}(\mathbf{k})t + i\zeta},$$

where

$$\frac{x_i}{t} = \frac{\partial \mathbf{W}(\mathbf{k})}{\partial k_i}.$$

For $\theta(\mathbf{x}, \mathbf{k}, t)$, $\mathbf{x} = (x_1, x_2, x_3)$ and

$$\omega = -\frac{\partial \theta}{\partial t},\ k_i = \frac{\partial \theta}{\partial x_i},\ \omega = \mathbf{W}(\mathbf{k}, \mathbf{x}, t).$$

Eliminating θ,

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0, \quad \frac{\partial k_i}{\partial x_j} - \frac{\partial k_j}{\partial x_i} = 0.$$

Extensions to Higher Dimensions

For $\omega = \mathbf{W}(\mathbf{k}, \mathbf{x}, t)$ varrying,

$$\frac{\partial k_i}{\partial t} + \frac{\partial \mathbf{W}}{\partial k_j}\frac{\partial k_j}{\partial x_i} = -\frac{\partial \mathbf{W}}{\partial x_i}.$$

$\partial k_j / \partial x_i = \partial k_i / \partial x_j$,

$$\frac{\partial k_i}{\partial t} + \frac{\partial k_i}{\partial x_j} = -\frac{\partial \mathbf{W}}{\partial x_i},\ C_j(\mathbf{k}, \mathbf{x}, t) = \frac{\partial \mathbf{W}}{\partial k_j},$$

C group velocity for some k_i,

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = \frac{\partial \mathbf{W}}{\partial x_i} \text{ on } \frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial\ \mathbf{W}}{\partial k_i}$$

called the characteristics form.
From above, we obtain the Hamiltonian-Jacobi equation

$$\frac{\partial \theta}{\partial t} + \mathbf{W}(\frac{\partial \theta}{\partial \mathbf{x}}, \mathbf{x}, t) = 0.$$

Energy Propagation

From

$$\varphi_{tt} - \alpha^2 \varphi_{xx} + \beta^2 \varphi = 0,$$

α and β constant,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varphi_t^2 + \frac{1}{2}\alpha^2\varphi_x^2 + \frac{1}{2}\beta^2\varphi^2\right) + \frac{\partial}{\partial x}\left(-\alpha^2\varphi_t\varphi_x\right) = 0,$$

and a slow wavetrain

$$\varphi \sim Re(Ae^{i\theta}) = a\cos(\theta + \eta), \; a = \mid A \mid, \; \eta = \arg A,$$

since

$$\frac{1}{2}\varphi_t^2 = \frac{1}{2}\omega^2 a^2 \sin^2(\theta + \eta),$$

Energy density

$$\frac{1}{2}(\omega^2 + \alpha^2 k^2)a^2 \sin^2(\theta + \eta) + \beta^2 a^2 \cos^2(\theta + \eta),$$

Energy Propagation

Flux

$$\alpha^2 \omega x a^2 \sin^2(\theta + \eta),$$

with slow varrying ω, k. We obtain

$$E_1 = \frac{1}{4}(\omega^2 + \alpha^2 k^2 + \beta^2) a^2, \; E_2 = \frac{1}{2}\alpha^2 \omega k a^2,$$

$\omega = \sqrt{\alpha^2 k^2 + \beta^2}, E_1 = \frac{1}{2}(\alpha^2 k^2 + \beta^2) a^2, \; E_2 = \frac{1}{2}\alpha^2 k \sqrt{\alpha^2 k^2 \beta^2 a^2}$
with $C(k)$ and E equation,

$$C(k) = \frac{\alpha^2 k}{\sqrt{\alpha^2 k^2 + \beta^2}}, \quad \Rightarrow E_2 = C(k) E_1,$$

$$\frac{\partial E_1}{\partial t} + \frac{\partial (C E_1)}{\partial t} = 0,$$

gives the differential form of energy.

Conclusion

- Basic notions and definition of Linear dispersive waves.
- Different modes propagate with different wave speeds.
- General solution and asymptotic analysis.
- Group and phase velocities.
- Energy propagation.

THANK YOU.