Water Waves

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Outline

- Introduction
- Equations for water waves
- Linear wave theory
- Classification of water waves
- Behavior near the front of the wavetrain
- Solution through the dispersion relation
- Conclusion

Introduction

The general idea of dispersive waves originated from the problems of water waves. The problems are of great interest in the Maritime and Offshore settings.

Figure: http://weblogs.sun-sentinel.com

Equations of water waves

Basic Assumptions

- we consider an inviscid incompressible fluid
- constant density ρ
- the spatial domain is given in (x_1, x_2, y) and
- the components of the velocity vector $\mathbf{u}$ by (u_1, u_2, v)
- $F = -\rho g \mathbf{j}$ be an external force on the fluid.
- assume the flow to be irrotational, $\omega = \nabla \times \mathbf{u} = 0$
- introduce a velocity potential φ such that $\mathbf{u} = \nabla \varphi$

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Linear wave theory

Plot of velocity potential with the velocity field

(velocity.avi)

Equations of water waves

then, the inviscid incompressible equations are

$$\nabla.\mathbf{u} = 0$$

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - g\mathbf{j}$$

so that

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}^2\right) + \omega \times \mathbf{u} = -\frac{1}{\rho}\nabla p - g\mathbf{j}$$

Equations of water waves

on integrating for

$$\mathbf{u} = \nabla\varphi$$

we have

$$\frac{p - p_0}{\rho} = -\varphi_t - \frac{1}{2}|\nabla\varphi|^2 - gy$$

Equations of water waves

Boundary conditions

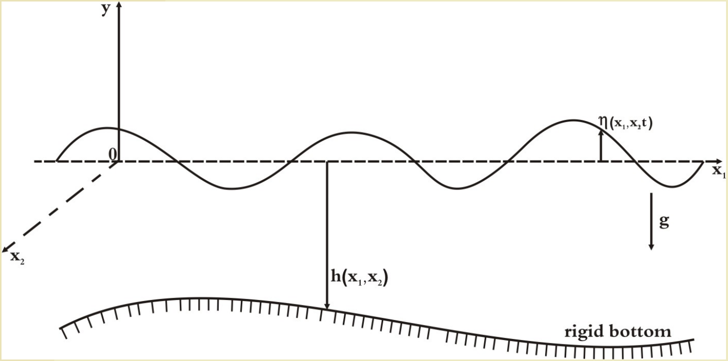

Figure: sketch of the flow domain and its boundaries

Equations of water waves

Define an interface

$$f(x_1, x_2, y, t) = 0$$

or for convenience as

$$y = \eta(x_1, x_2, t) \quad \text{such that } f(x_1, x_2, y, t) \equiv \eta(x_1, x_2, t) - y$$

Kinematic free surface condition

$$\frac{D\eta}{Dt} = \eta_t + u_1 \eta_{x_1} + u_2 \eta_{x_2} = v$$

Equations of water waves

and we obtain

Dynamic free surface condition

$$p = p_0$$

Boundary conditions at the free surface

$$\begin{aligned} \eta_t + \varphi_{x_1}\eta_{x_1} + \varphi_{x_2}\eta_{x_2} - \varphi_y &= 0, \qquad \text{on } y = \eta, \\ \varphi_t + \frac{1}{2}\left|\nabla\varphi\right|^2 + g\eta &= 0, \qquad \text{on } y = \eta \end{aligned}$$

Equations of water waves

Consider the bottom

$$y = -h_0(x_1, x_2)$$

we obtain Kinematic bottom boundary condition

$$\varphi_{x_1} h_{0_{x_1}} + \varphi_{x_2} h_{0_{x_2}} + \varphi_y = 0, \qquad \text{on } y = -h_0$$

and for a horizontal flat bottom,

$$\varphi_y = 0, \qquad \text{on } y = -h_0$$

Linear wave theory

the linearized free surface conditions are

$$\begin{aligned} \eta_t &= \varphi_y, &\quad \text{on } y = \eta, \\ \varphi_t + g\eta &= 0, &\quad \text{on } y = \eta. \end{aligned}$$

$$\varphi_{tt} + g\varphi_y = 0, \qquad \text{on } y = 0$$

the surface elevation is

$$\eta = -\frac{1}{g}\varphi_t(x_1, x_2, 0, t)$$

Linear wave theory

The linearized formulation

$$\begin{aligned} \varphi_{x_1x_1} + \varphi_{x_2x_2} + \varphi_{yy} &= 0, &\qquad &\text{on } -h_0 < y < 0, \\ \varphi_{tt} + g\varphi_y &= 0, &\qquad &\text{on } y = 0, \\ \varphi_{x_1} h_{0_{x_1}} + \varphi_{x_2} h_{0_{x_2}} + \varphi_y &= 0, &\qquad &\text{on } y = -h_0 \end{aligned}$$

Linear wave theory

Analytic solution of the wave problem

Form of solution for water waves

$$\eta = A \exp\{i(\kappa x - \omega t)\}$$

where *A* is the Amplitude of the wave

$$\varphi = Y(y) \exp\{i(\kappa x - \omega t)\}$$

Linear wave theory

The Laplace equation

$$\varphi_{xx} + \varphi_{yy} = 0$$

Method:Separation of Variables

- In 2-D, we define

$$\varphi(x, y, t) = X(x)Y(y)T(t)$$

- divide by X(x)Y(y)T(t) such that

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\kappa^2$$

Linear wave theory

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Linear wave theory

where κ^2 is the separation constant.
We obtain

$$X'' + \kappa^2 X = 0$$
$$Y'' - \kappa^2 Y = 0$$

solving these equations, we obtain

$$X = B\cos\kappa x + D\sin\kappa x$$
$$Y = Ee^{\kappa y} + Ge^{-\kappa y}$$

The solution is given by

$$\varphi(x, y, t) = (B\cos\kappa x + D\sin\kappa x)(Ee^{\kappa y} + Ge^{-\kappa y})T(t)$$

Linear wave theory

We choose for $T(t)$

$$\cos\omega t \quad \text{or} \quad \sin\omega t$$

where

$$\omega = \frac{2\pi}{T}$$

and arrive at 4 possible solutions, periodic in x and t

$$\begin{aligned}
\varphi_1 &= A_1 Y(y) \cos\kappa x \cos\omega t\\
\varphi_2 &= A_2 Y(y) \sin\kappa x \sin\omega t\\
\varphi_3 &= A_3 Y(y) \sin\kappa x \cos\omega t\\
\varphi_4 &= A_4 Y(y) \cos\kappa x \sin\omega t
\end{aligned}$$

Linear wave theory

The first equation gives the following solution

$$\varphi_1 = \frac{-gA}{\omega}\frac{\cosh\kappa(h_0+y)}{\cosh\kappa h_0}\cos\kappa x\cos\omega t \quad \text{and}$$

$$\eta_1 = A\cos\kappa x\sin\omega t$$

The dispersion relation is

$$W(\kappa) = \omega^2 = g\kappa\tanh\kappa h_0$$

Classification of water waves

Water waves are classified into three main categories:

- Shallow water or long waves, if

$$\frac{h_0}{\lambda} < \frac{1}{20}$$

- Deep water waves, if

$$\frac{h_0}{\lambda} > \frac{1}{2}$$

- Intermediate water waves, if

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Classification of water waves

The dispersion relation can be written as

$$C^2 = \frac{g}{\kappa} \tanh \kappa h_0$$

For shallow water waves,

$$C^2 = gh_0$$

and for deep water waves,

$$C^2 = \frac{g}{\kappa}$$

where C is the phase velocity.

Classification of water waves

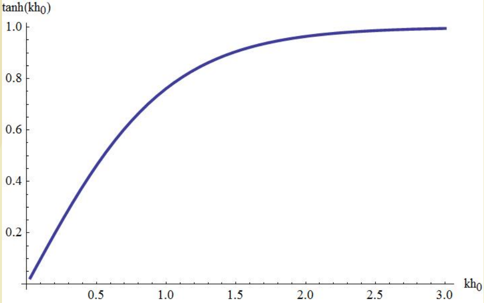

Figure: $\tanh(\kappa h_0) \simeq 1$ for $\kappa h_0 \to \infty$ and $\tanh(\kappa h_0) \simeq \kappa h_0$ for $\kappa h_0 \to 0$

Classification of water waves

$$C_g = \frac{d(C\kappa)}{d\kappa} = \frac{1}{2} C \left(1 + \frac{2\kappa h_0}{\sinh(2\kappa h_0)} \right)$$

$$\frac{2\kappa h_0}{\sinh(2\kappa h_0)} \to 0 \quad \text{for} \quad \kappa h_0 \to \infty$$

and

$$\frac{2\kappa h_0}{\sinh(2\kappa h_0)} \to 1 \quad \text{for} \quad \kappa h_0 \to 0$$

For Deep water waves,

$$\frac{C_g}{C} \sim \frac{1}{2} \quad \text{for} \quad \kappa h_0 \to \infty,$$

For Shallow water waves

$$\frac{C_g}{C} \sim 1 \quad \text{for} \quad \kappa h_0 \to 0$$

Behavior near the front of the wavetrain

Initial value problem

We consider the fluid initially at rest with $\varphi = 0$ then

$$\eta_t = 0 \qquad \text{such that}$$
$$\eta(x,0) = \eta_0(x), \qquad t = 0$$

The general solution is by method of Fourier integral

$$\eta(x,t) = \int_{-\infty}^{\infty} F(\kappa) e^{i(\kappa x - Wt)} \mathrm{d}\kappa + \int_{-\infty}^{\infty} F(\kappa) e^{i(\kappa x + Wt)} \mathrm{d}\kappa$$

Behavior near the front of the wavetrain

where $F(\kappa)$ is the Fourier transform of $\frac{1}{2}\eta_0(x)$ given by

$$F(\kappa) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \eta_0(x) e^{-i\kappa x} \mathrm{d}x$$

Using the asymptotic results, we have

$$\eta \sim 2\mathcal{R}\left(F(k)\sqrt{\frac{2\pi}{t|W''(k)|}} e^{ikx - iW(k)t - \frac{i\pi}{4} sgn(W''(k))} \right),$$

for $t \to \infty$, $\frac{x}{t} > 0$

Behavior near the front of the wavetrain

For finite depth, $W \sim \kappa\sqrt{gh_0}$ such that

$$C_g(\kappa) = W'(\kappa) \rightarrow \sqrt{gh_0}$$

and $W''(\kappa) \rightarrow 0$ as $\kappa h_0 \rightarrow 0$

Behavior near the front of the wavetrain

What are our expections?

- We expect an amplitude decay of

$$\eta \propto t^{-\frac{1}{3}}$$

- and a solution that is uniformly valid through the whole transition region.

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Behavior near the front of the wavetrain

How do we achieve this?

- We expand $W(\kappa)$ in the Fourier integral about $\kappa = 0$ so that

$$W(\kappa) \sim c_0\kappa - \gamma\kappa^3 + \ldots$$

- and also $F(\kappa)$, retaining only the first term,

$$F(0) = \frac{1}{4\pi}$$

where $c_0 = \sqrt{gh_0}$ and $\gamma = \frac{1}{6}h_0^2\sqrt{gh_0}$

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Behavior near the front of the wavetrain

finally, we have

$$\eta \sim \eta_f = \frac{1}{4\pi} \int_{-\infty}^{\infty} \exp\left\{ i\kappa(x - c_0 t) + i\gamma\kappa^3 t \right\} \mathrm{d}\kappa$$

Using the standard Airy integral

$$Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{ i(sz + \frac{1}{3}s^3) \right\} \mathrm{d}s$$

$$\eta_f = \frac{1}{2(3\gamma t)^{\frac{1}{3}}} Ai\left\{ \frac{x - c_0 t}{(3\gamma t)^{\frac{1}{3}}} \right\}$$

Behavior near the front of the wavetrain

Exact solution of the wave moving to the right

(frame00.avi)

Behavior near the front of the wavetrain

The Airy function has the following asymptotic behavior

$$Ai(z) \sim \begin{cases} \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp\left(-\frac{2}{3} z^{\frac{3}{2}}\right), & z \to +\infty, \\ \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \sin\left(\frac{2}{3} |z|^{\frac{3}{2}} + \frac{\pi}{4}\right), & z \to -\infty. \end{cases}$$

where $z = \frac{(x - c_0 t)}{(3\gamma t)^{\frac{1}{3}}}$

Behavior near the front of the wavetrain

Asymptotic Solution

(asymp.avi)

Solution through the dispersion relation

The dispersion relation gives a correspondence to the linearized Korteweg DeVries equation

$$\eta_t + c_0\eta_x + \gamma\eta_{xxx} = 0$$

on solving the equation with the initial condition

$$\eta_0(x) = \frac{1}{2}\delta(x)$$

we obtain the solution

$$\eta(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \exp\left\{ i\kappa(x - c_0 t) + i\gamma\kappa^3 t \right\} \mathrm{d}\kappa$$

Conclusion

- The asymptotic theory gives a better description of the behavior of the solution.
- In the linear theory, the long waves can be described by the linearized Korteweg DeVries equation.
- The validity of the linear theory near the front of the wavetrain is nonuniform.

Thank you