Second-generation wavelets on finite intervals

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Overview of the talk

1. Quick review of some concepts

2. The Lifting scheme

3. Numerical examples of Lifting, and some problems

4. Wavelets on finite intervals
   (second-generation wavelets)

5. Numerical examples of interval wavelets

6. Other second-generation wavelets

Quick review of some concepts

- Riesz bases and their dual
- Multi-resolution analysis
- Wavelets and scaling functions
- Discrete Wavelet Transform (DWT)

Riesz bases

Suppose we have a Hilbert space $\mathcal{H}$ and a (countably infinite) sequence of vectors $\{\phi_n\} \subset \mathcal{H}$. Define the operator $\Phi : \ell^2 \to \mathcal{H}$ as

$$\Phi(\{\alpha_n\}) := \sum_{n=0}^{\infty} \alpha_n \phi_n. \tag{1}$$

Then $\{\phi_n\}$ is a *Riesz basis* of $\mathcal{H}$ if and only if:

1. The series (1) converges for all $\{\alpha_n\} \in \ell^2$.

2. The operator Φ is bounded.

3. The inverse $\Phi^{-1} : \mathcal{H} \to \ell^2$ exists.

4. The inverse Φ^{-1} is also bounded.

What does it mean?

This means that a Riesz basis gives us a way to map the abstract space $\mathcal{H}$ one-to-one on the concrete space ℓ^2. Convergence of a sequence in ℓ^2 implies convergence of the Φ-transformed sequence in $\mathcal{H}$, and vice versa.

Note that if we have a bounded operator $\Psi : \ell^2 \to \mathcal{H}$ with bounded inverse Ψ^{-1}, then we can find back the accompanying Riesz basis $\{\psi_n\}$ by

$$\psi_n := \Psi(e_n), \quad \text{where } e_n(m) = \delta_{nm}. \qquad (2)$$

Dual basis

The dual basis $\{\tilde{\phi}_n\}$ of a Riesz basis $\{\phi\}_n$ satisfies the following equation.

$$(\phi_i, \tilde{\phi}_j) = \delta_{ij}. \tag{3}$$

The corresponding operators Φ and $\tilde{\Phi}$ satisfy the relation

$$\tilde{\Phi} = \left(\Phi^{-1}\right)^*. \tag{4}$$

If $\{\phi_n\} = \{\tilde{\phi}_n\}$ (which is true iff $\Phi = \tilde{\Phi}$), then we say that $\{\phi_n\}$ is an orthonormal basis.

Note that Φ is in that case an orthogonal operator, since

$$\Phi^* = \Phi^{-1}. \tag{5}$$

Multi-resolution analysis

Consider the function space $L^2(\mathbb{R})$.

A *Multi-resolution analysis* consists of a sequence of linear subspaces $\{V_n\}$, $n \in \mathbb{Z}$, with the following properties.

1. $V_n = V_{n-1} \oplus W_{n-1}$, for some $\{W_n\}$.
2. $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$.
3. $\bigcap_{n\in\mathbb{Z}} V_n = \{\mathbf{0}\}$.
4. $f(t) \in V_n \Leftrightarrow f(2t) \in V_{n+1}$.
5. $f(t) \in V_0 \Leftrightarrow f(t+1) \in V_0$.

Analysis and synthesis

Analysis

$$\begin{tikzcd}
V_3 \arrow[r] \arrow[dr] & V_2 \arrow[r] \arrow[dr] & V_1 \arrow[r] \arrow[dr] & V_0 \\
 & W_2 & W_1 & W_0
\end{tikzcd}$$

Synthesis

$$\begin{tikzcd}
V_3 & V_2 \arrow[l] & V_1 \arrow[l] & V_0 \arrow[l] \\
 & W_2 \arrow[ul] & W_1 \arrow[ul] & W_0 \arrow[ul]
\end{tikzcd}$$

Wavelets and scaling functions

1. A Riesz basis of V_i is formed by the scaling functions $\{\phi_{i,n}(t)\}_n$.

2. A Riesz basis of the detail space W_i is formed by the wavelets $\{\psi_{i,n}(t)\}_n$.

3. All scaling functions are scaled, translated versions of the father wavelet $\phi(t)$.

4. All wavelet functions are scaled, translated versions of the mother wavelet $\psi(t)$.

5. Scaling functions and wavelets are localised, i.e. $\lim_{t\to\pm\infty} \phi_{i,n}(t), \psi_{i,n}(t) = 0$.

6. $\int_{-\infty}^{\infty} \psi_{i,n}(t) = 0$, i.e. wavelets have zero average.

Discrete Wavelet Transform

We have a $f \in V_i$, represented by coefficients $\{v_i(n)\}_n$.

$$f(t) = \sum_{n=-\infty}^{\infty} v_i(n)\phi_{i,n}(t). \tag{6}$$

Since $V_i = V_{i-1} \oplus W_{i-1}$, we can write $f(t) = g(t) + h(t)$, with $g \in V_{i-1}$ and $h \in W_{i-1}$.

$$g(t) = \sum_{n=-\infty}^{\infty} v_{i-1}(n)\phi_{i-1,n}(t), \tag{7}$$

$$h(t) = \sum_{n=-\infty}^{\infty} w_{i-1}(n)\psi_{i-1,n}(t). \tag{8}$$

The linear operation that produces $\{v_{i-1}(n)\}_n$ and $\{w_{i-1}(n)\}_n$ from $\{v_i(n)\}$ is called the *Discrete Wavelet Transform* (DWT).

Its inverse is called the *Inverse Discrete Wavelet Transform* (IDWT).

The Lifting Scheme

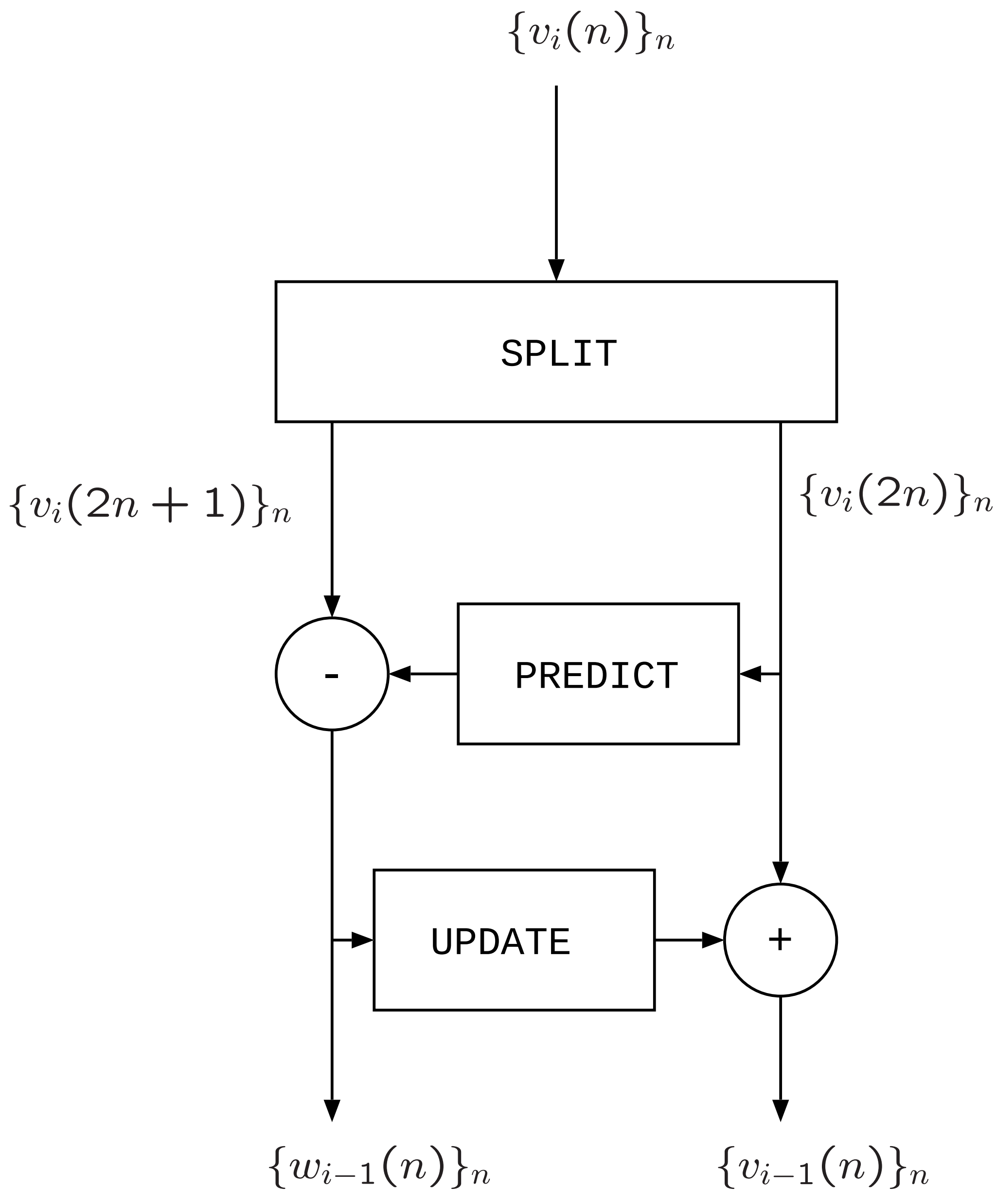
The lifting scheme is a method for designing wavelet transforms.

First we design a Discrete Wavelet Transform (DWT) directly, without any reference to scaling functions $\phi_{i,n}$ or wavelets $\psi_{i,n}$.

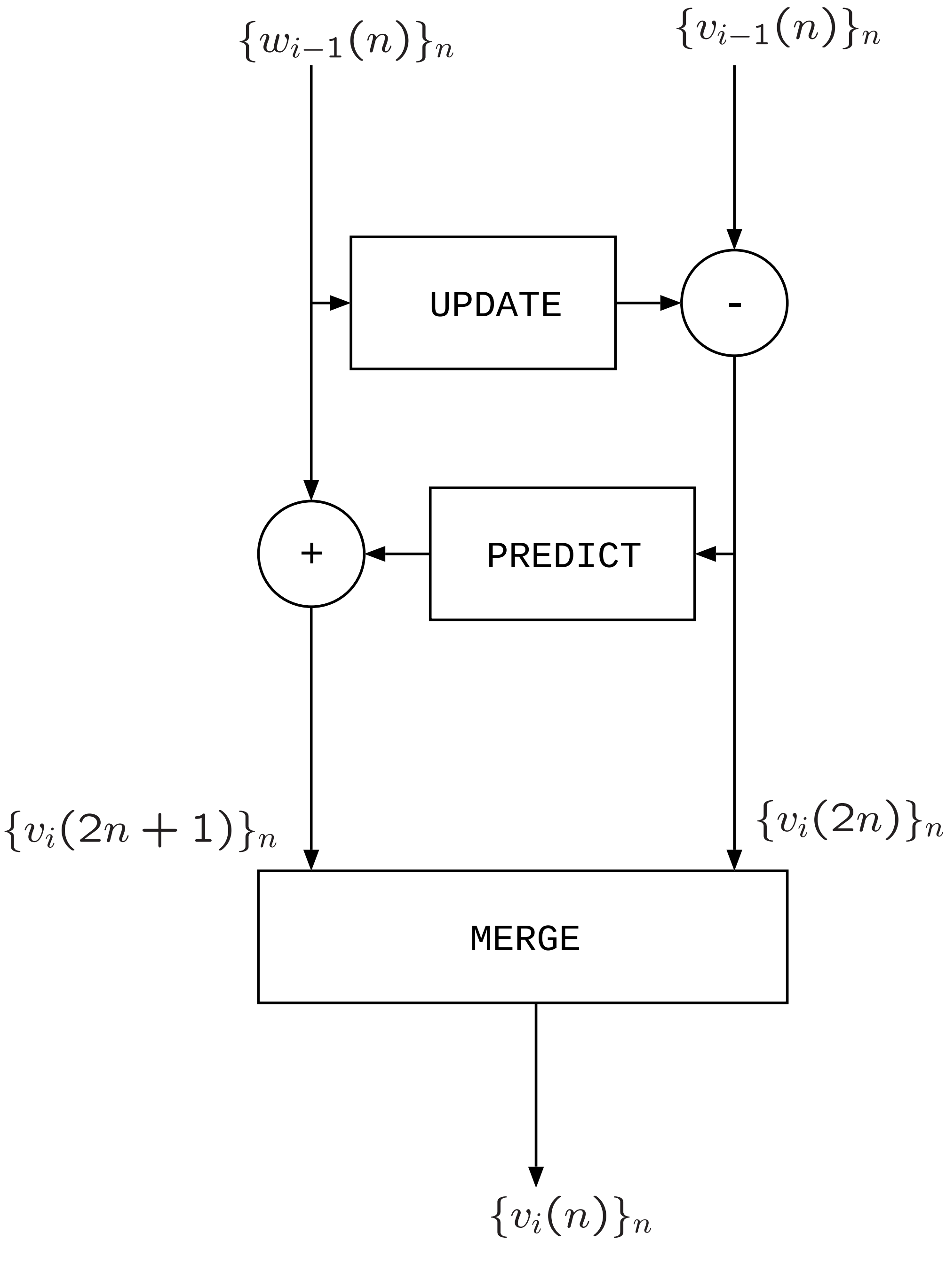
Then we find the scaling functions and wavelets belonging to this DWT. *Note that we don't need these for most applications.*

Exactly the opposite of what you might expect!

Lifting scheme: DWT



Lifting scheme: IDWT



Discrete Wavelet Transform:

$$\{v_i(2n)\}_n, \{v_i(2n+1)\}_n = \mathsf{SPLIT}(\{v_i(n)\}_n),$$

$$\{w_{i-1}(n)\}_n = \{v_i(2n+1)\}_n - \mathsf{P}(\{v_i(2n)\}_n),$$

$$\{v_{i-1}(n)\}_n = \{v_i(2n)\}_n + \mathsf{U}(\{w_{i-1}(n)\}_n).$$

Inverse Discrete Wavelet Transform:

$$\{v_i(2n)\}_n = \{v_{i-1}(n)\}_n - \mathsf{U}(\{w_{i-1}(n)\}_n).$$

$$\{v_i(2n+1)\}_n = \{w_{i-1}(n)\}_n + \mathsf{P}(\{v_i(2n)\}_n),$$

$$\{v_i(n)\}_n = \mathsf{MERGE}(\{v_i(2n)\}_n, \{v_i(2n+1)\}_n).$$

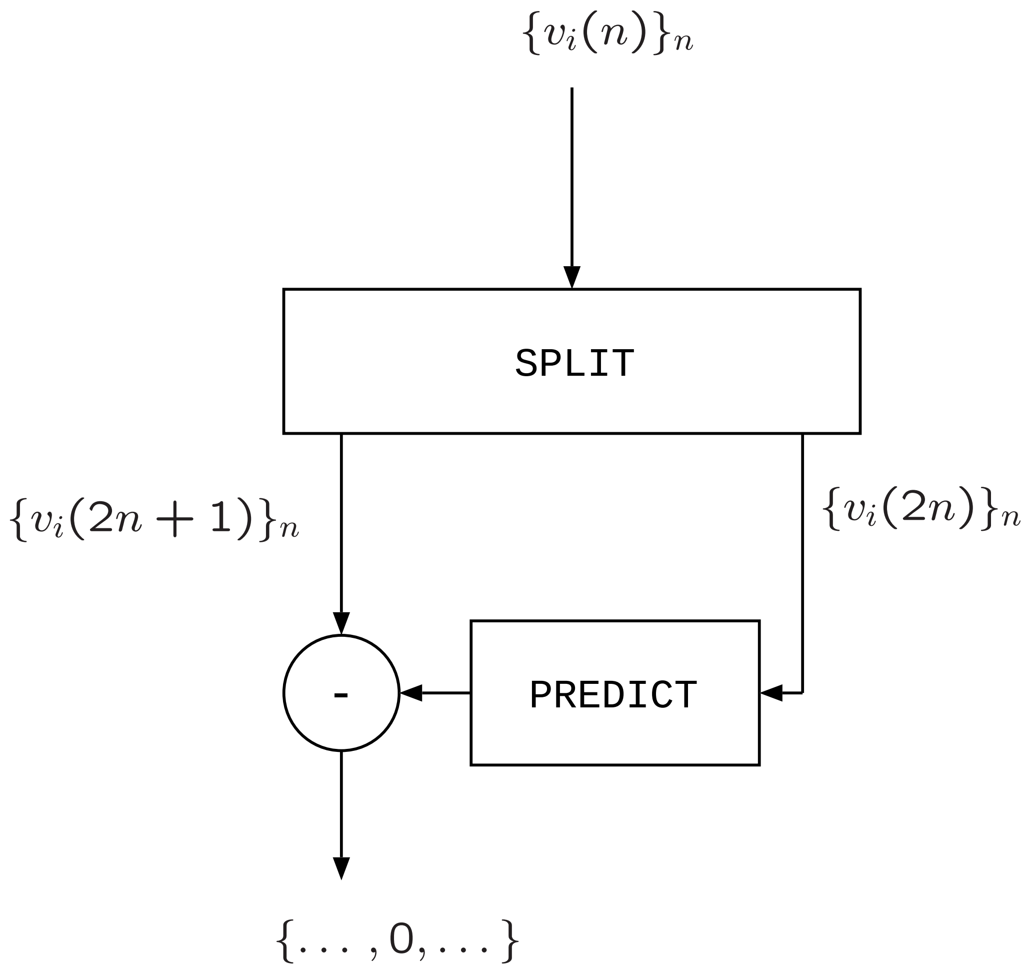
Choosing P and U

- P is selected so that certain classes of functions can be represented exactly (lead to 0 detail coefficients) in any space V_i. E.g. constant functions or polynoms up to some degree k.

- U is selected so that the contribution of the detail coefficients $\{w_{i-1}(n)\}_n$ to the signal $\{v_i(n)\}_n$ has zero average.
  More generally, one requires that the first k moments are 0.

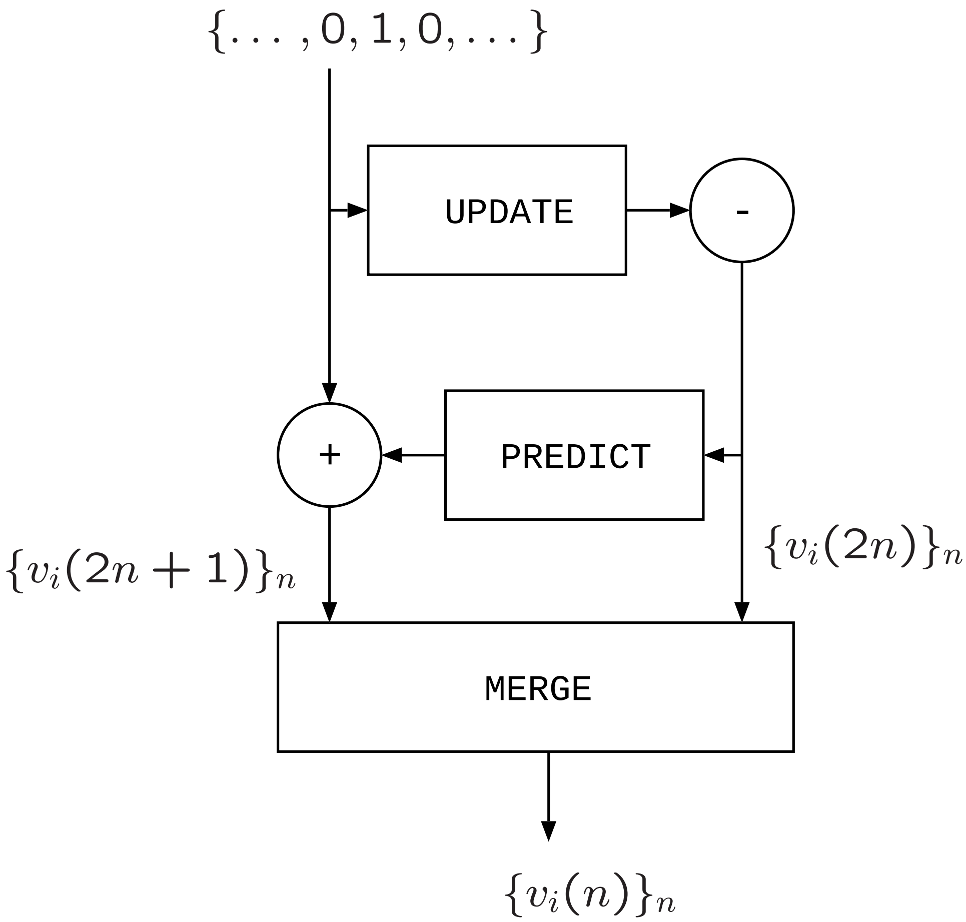
$$\sum_{n=-\infty}^{\infty} v_i(n)n^p = 0, \quad \text{for } p = 0, \ldots, k-1. \tag{9}$$

Designing the predict step



We design the operator P so that for certain classes of inputs $\{v_i(n)\}_n$, the detail signal becomes exactly 0.

Designing the update step



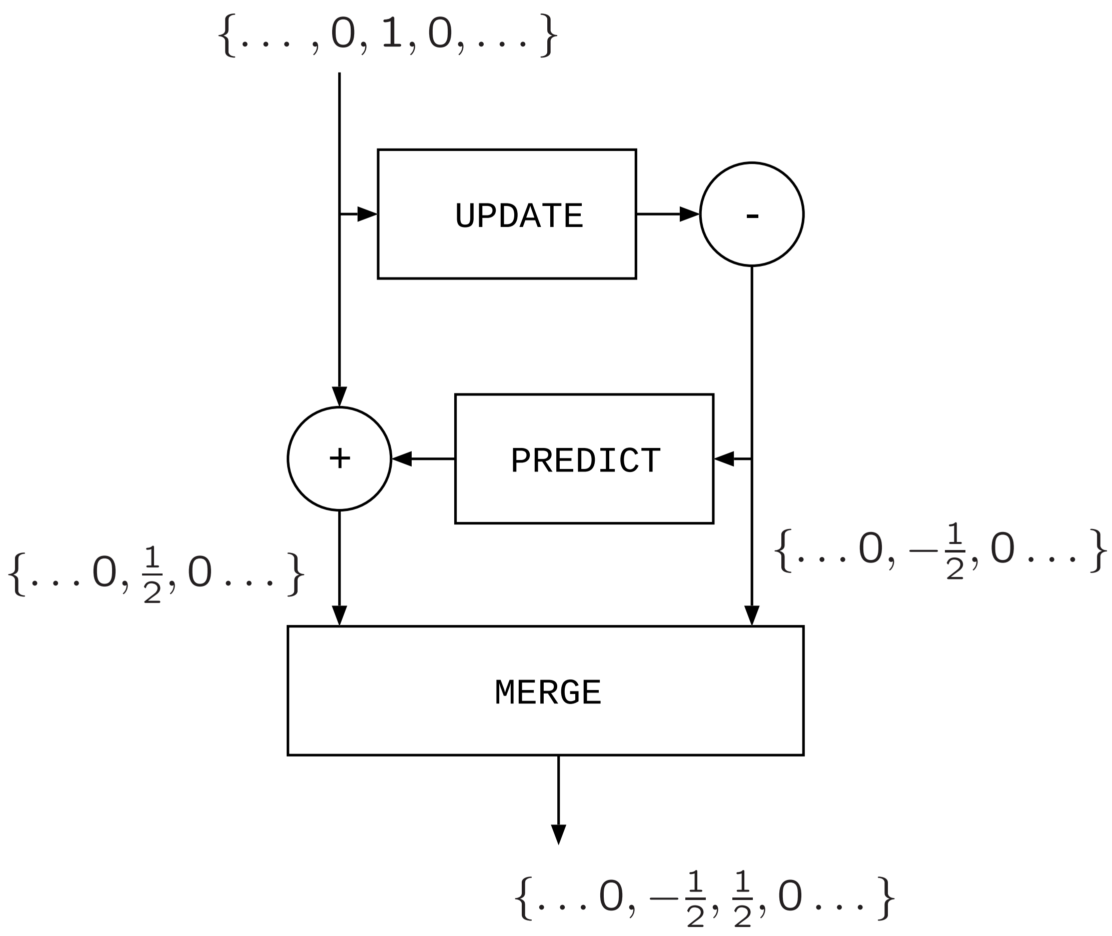
We design the operator U so that if a δ-pulse is put on the detail wire and 0 on the other wire, we obtain a $\{v_i(n)\}_n$ with 0 average.

Example: Haar wavelet

- $\mathrm{P}(\{v_i(2n)\}_n) = \{v_i(2n)\}_n$,
  i.e. just the identity.

- $\mathrm{U}(\{w_{i-1}(n)\}_n) = \{\frac{1}{2}w_{i-1}(n))\}_n$,
  i.e. multiplication by $\frac{1}{2}$.

Haar wavelet update step



Example: Cohen-Daubechies-Feauveau biorthogonal $(2, 2)$ wavelet

- $\mathsf{P}(\{v_i(2n)\}_n) = \{\frac{1}{2}(v_i(2n) + v_i(2n+2))\}_n$,

- $\mathsf{U}(\{w_{i-1}(n)\}_n) = \{\frac{1}{4}(w_{i-1}(n-1)+w_{i-1}(n))\}_n$.

Computing the scaling function

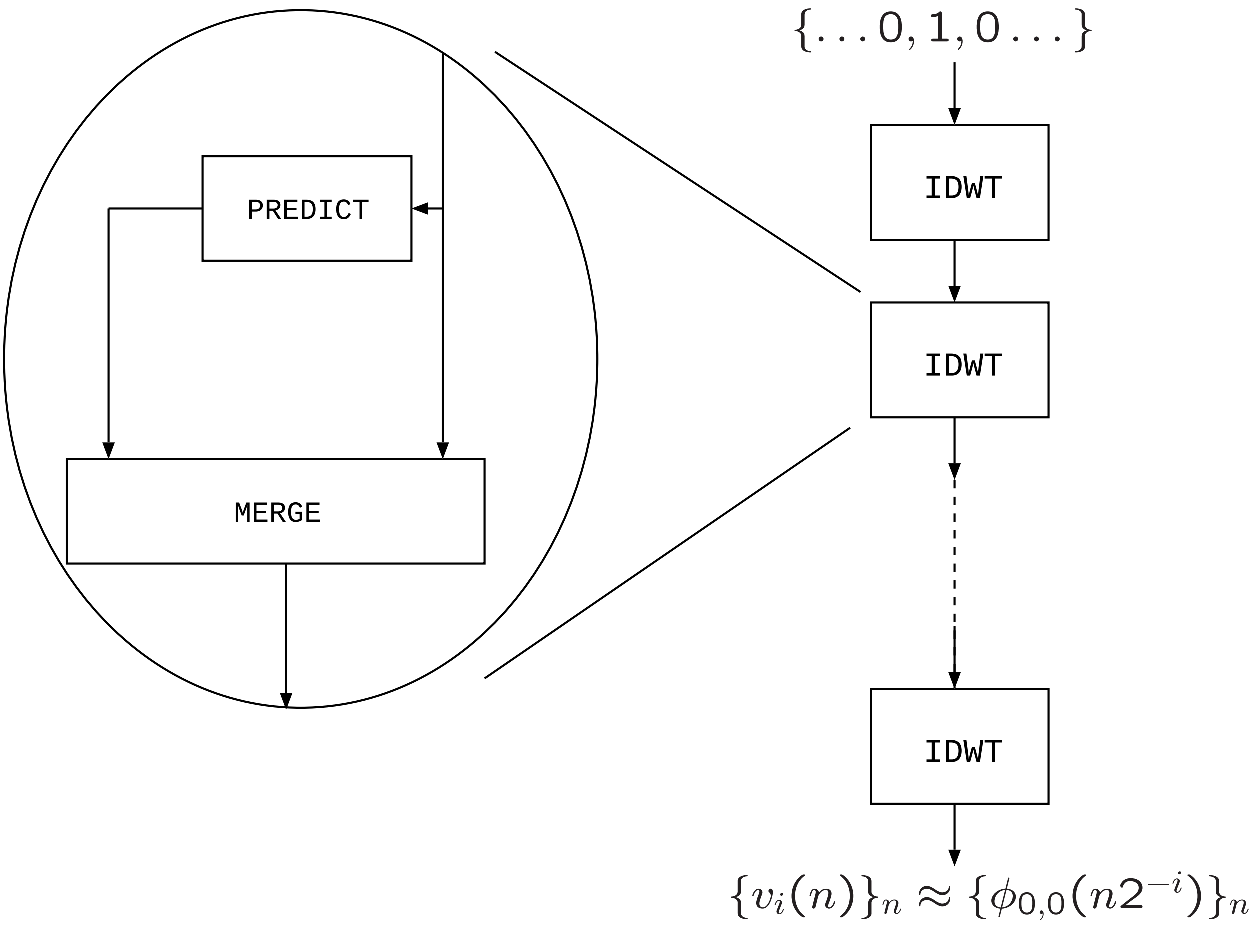
In the space V_0, the basic scaling function $\phi_{0,0}$ (the father wavelet) is represented by the coefficients

$$\{v_0(n)\}_n = \{\ldots, 0, 1, 0 \ldots\} = \delta_n.$$

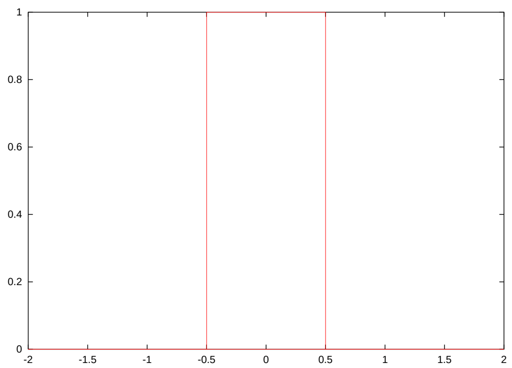
Using the Inverse DWT (with all detail coefficients $w_i(n) = 0$), we can compute the representation of the scaling function in any space V_i, $i > 0$.

For $i \to \infty$, the coefficients $\{v_i(n)\}_n$ approach the sampling $\{\phi_{0,0}(n2^{-i})\}_n$.

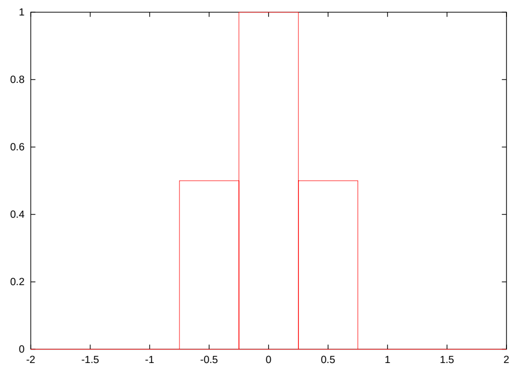
The cascade algorithm



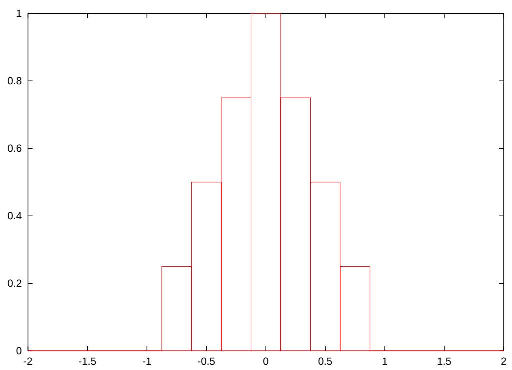
Example: CDF-$(2, 2)$ scaling function



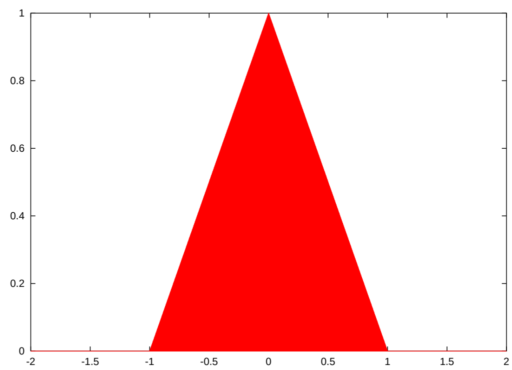
Iteration 0



Iteration 1



Iteration 2



Iteration 8

Computing the wavelet

In the detail space W_0, the basic wavelet $\psi_{0,0}$ (the mother wavelet) is represented by the coefficients

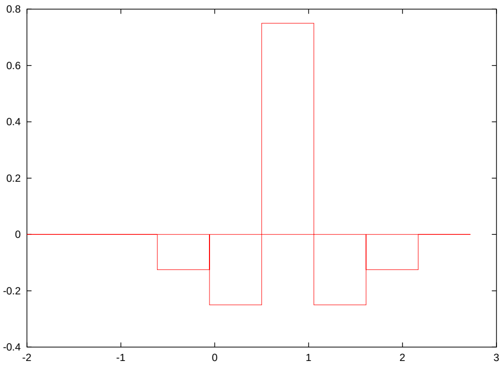
$$\{w_0(n)\}_n = \{\ldots, 0, 1, 0 \ldots\} = \delta_n.$$

Using the Inverse DWT (with all coefficients $v_0(n) = 0$), we can compute the representation of the scaling function in V_1.

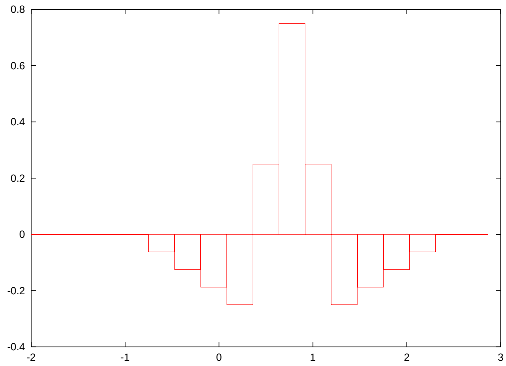
Then we can again use the cascade algorithm to find the representation in any space V_i, $i > 1$.

For $i \to \infty$, the coefficients $\{v_i(n)\}_n$ approach the sampling $\{\psi_{0,0}(n2^{-i})\}_n$.

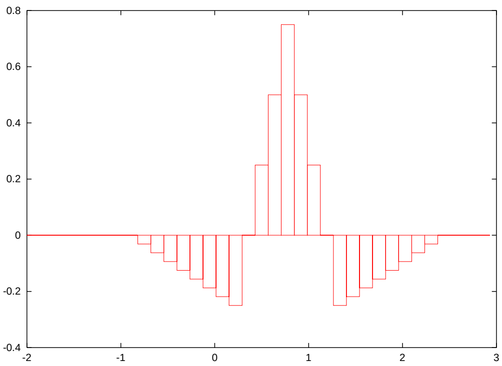
Example: CDF-$(2,2)$ wavelet



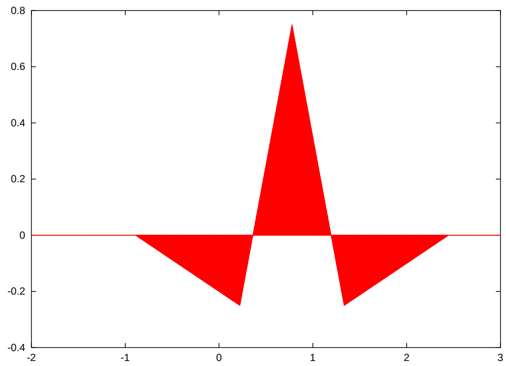
Iteration 0



Iteration 1

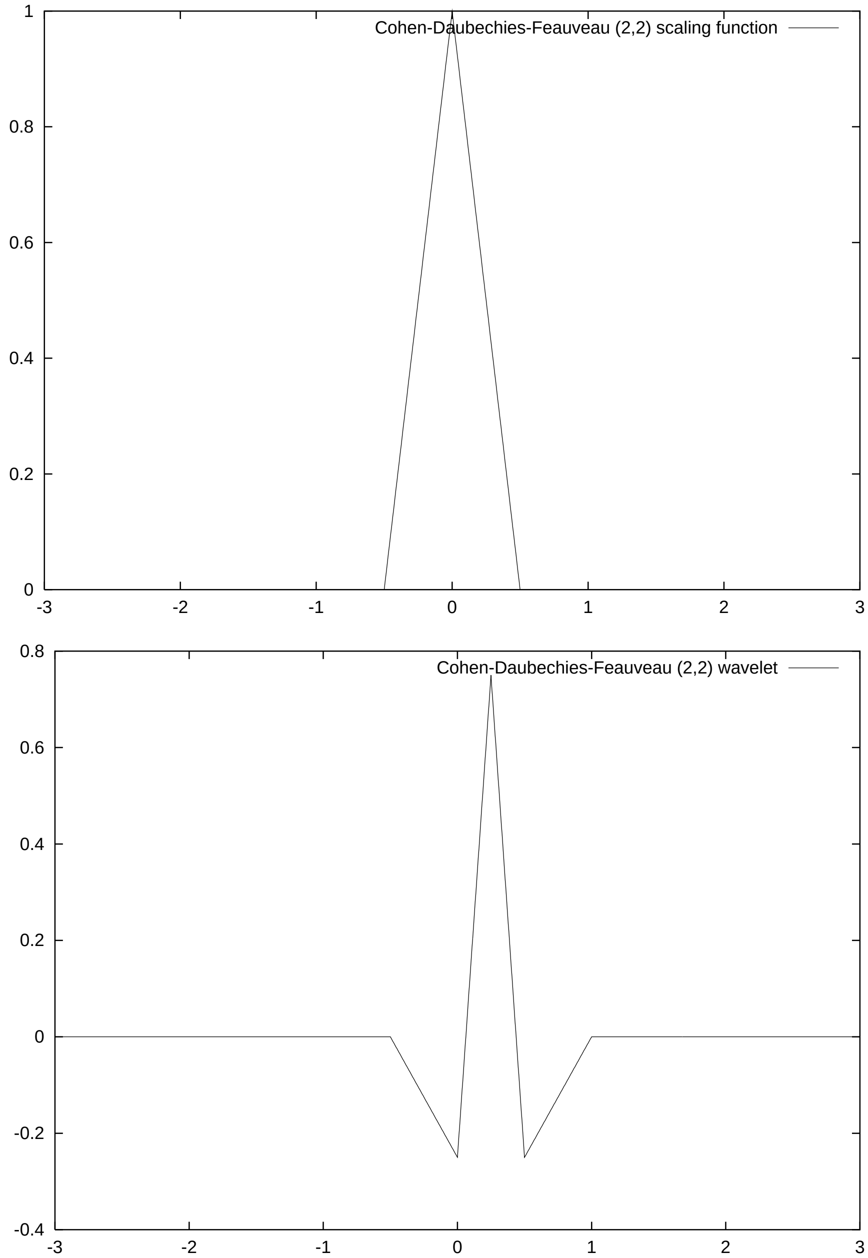


Iteration 2



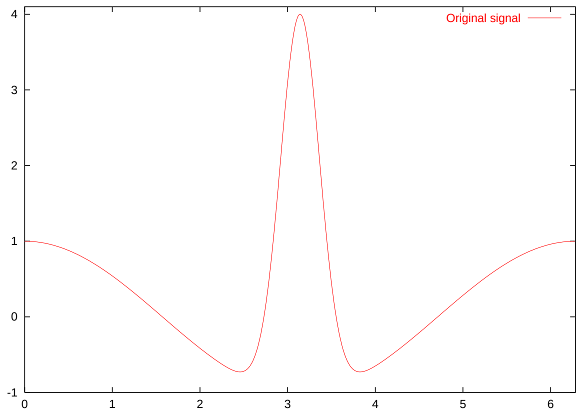
Iteration 8

CDF-$(2, 2)$

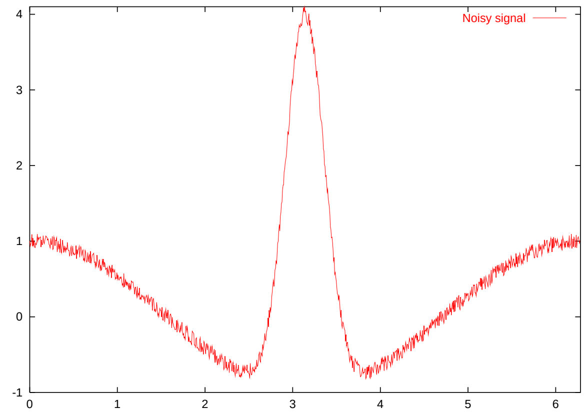


Cohen-Daubechies-Feauveau (2,2) scaling function
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3
Cohen-Daubechies-Feauveau (2,2) wavelet
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-3
-2
-1
0
1
2
3

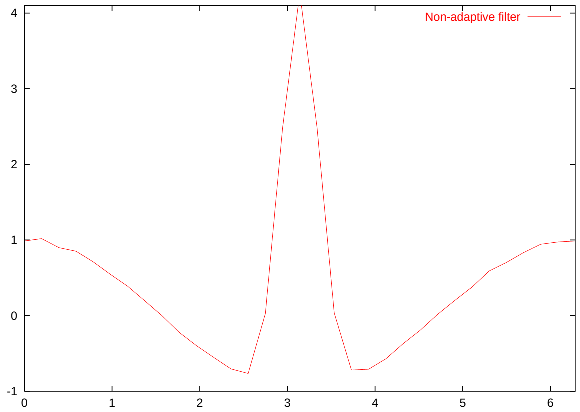
CDF-$(2,2)$ for de-noising



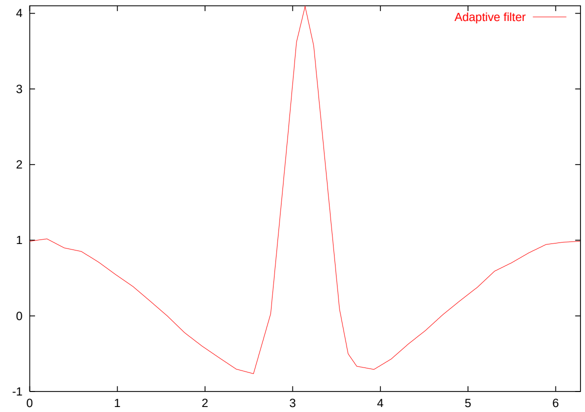
Original signal



Noisy signal

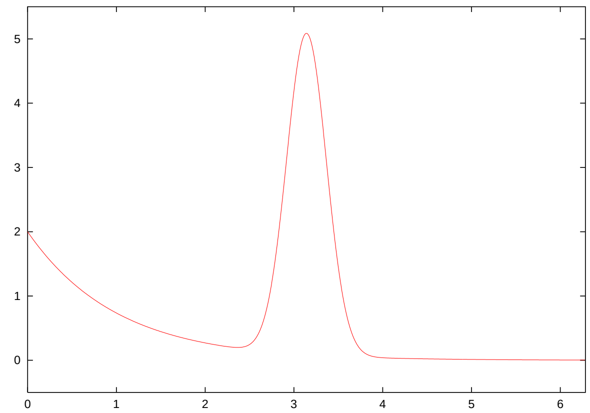


Non-adaptive filter

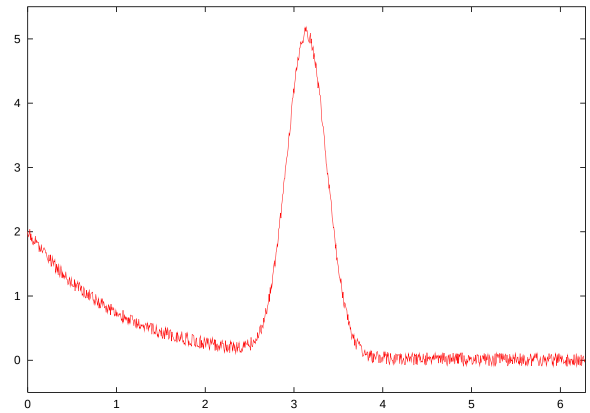


Adaptive filter

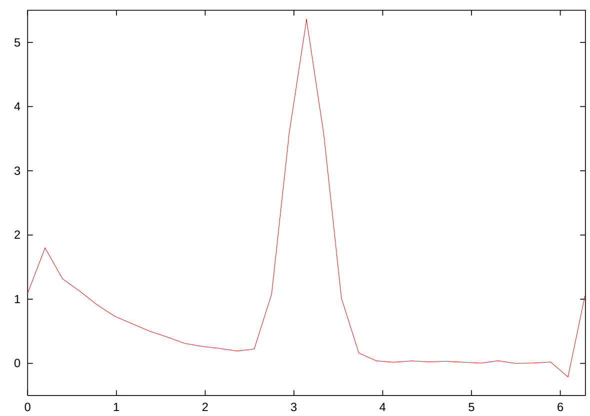
CDF-$(2, 2)$ for de-noising (2)



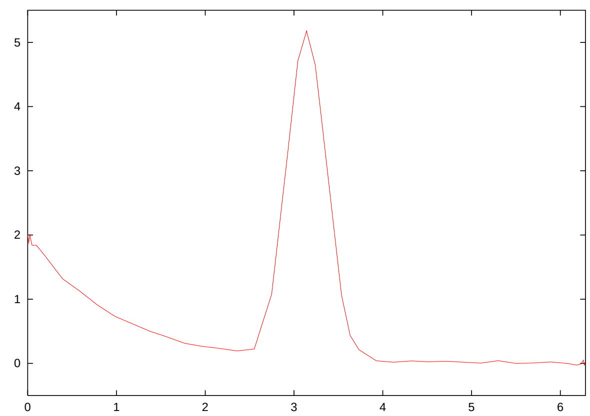
Original signal



Noisy signal

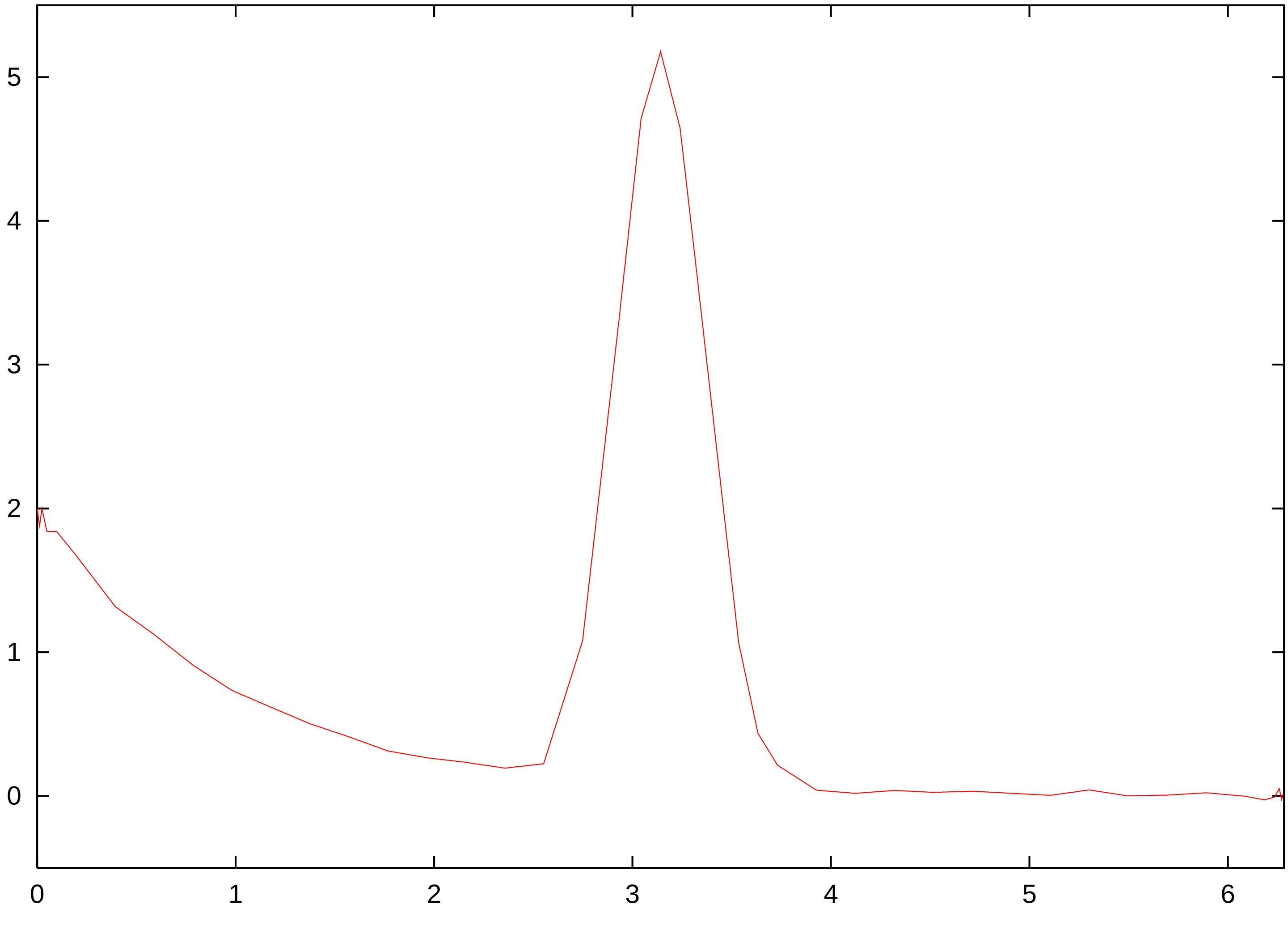


Non-adaptive filter



Adaptive filter

CDF-$(2, 2)$, non-periodic signal, adaptive filter



Wavelets on finite intervals

- Away from the boundary, wavelets and scaling function remain the same.

- Near the boundary, we introduce special wavelets and scaling functions.

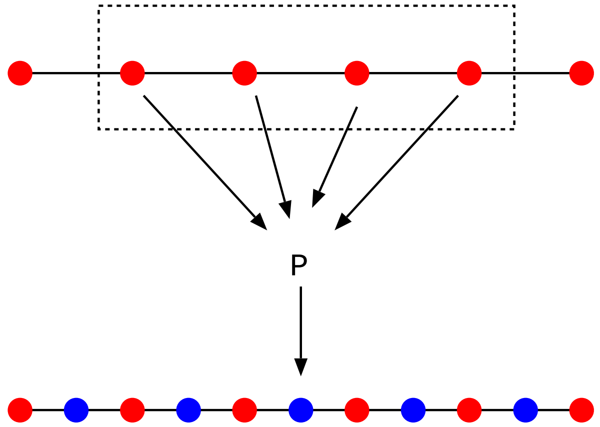
- We again use the lifting scheme to find the DWT first, and then compute the wavelets and scaling functions by using the cascade algorithm.

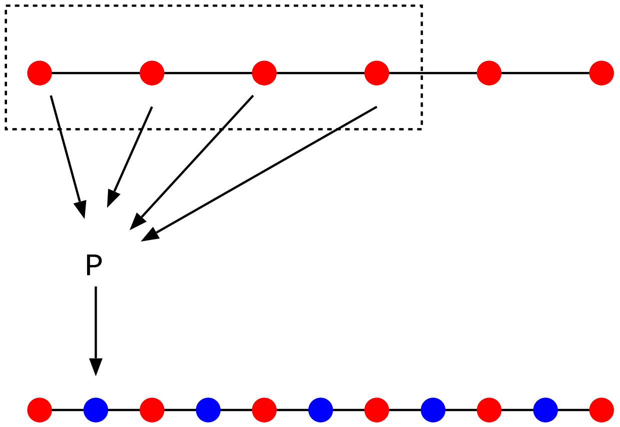
- It is *no longer true* that all wavelets and scaling functions are translated, dilated versions of the mother and father wavelet. (Orphan wavelets)

- Interval wavelets are a special case of *second-generation wavelets*.

Predict for interval wavelets

Away from the boundary, P works as before.

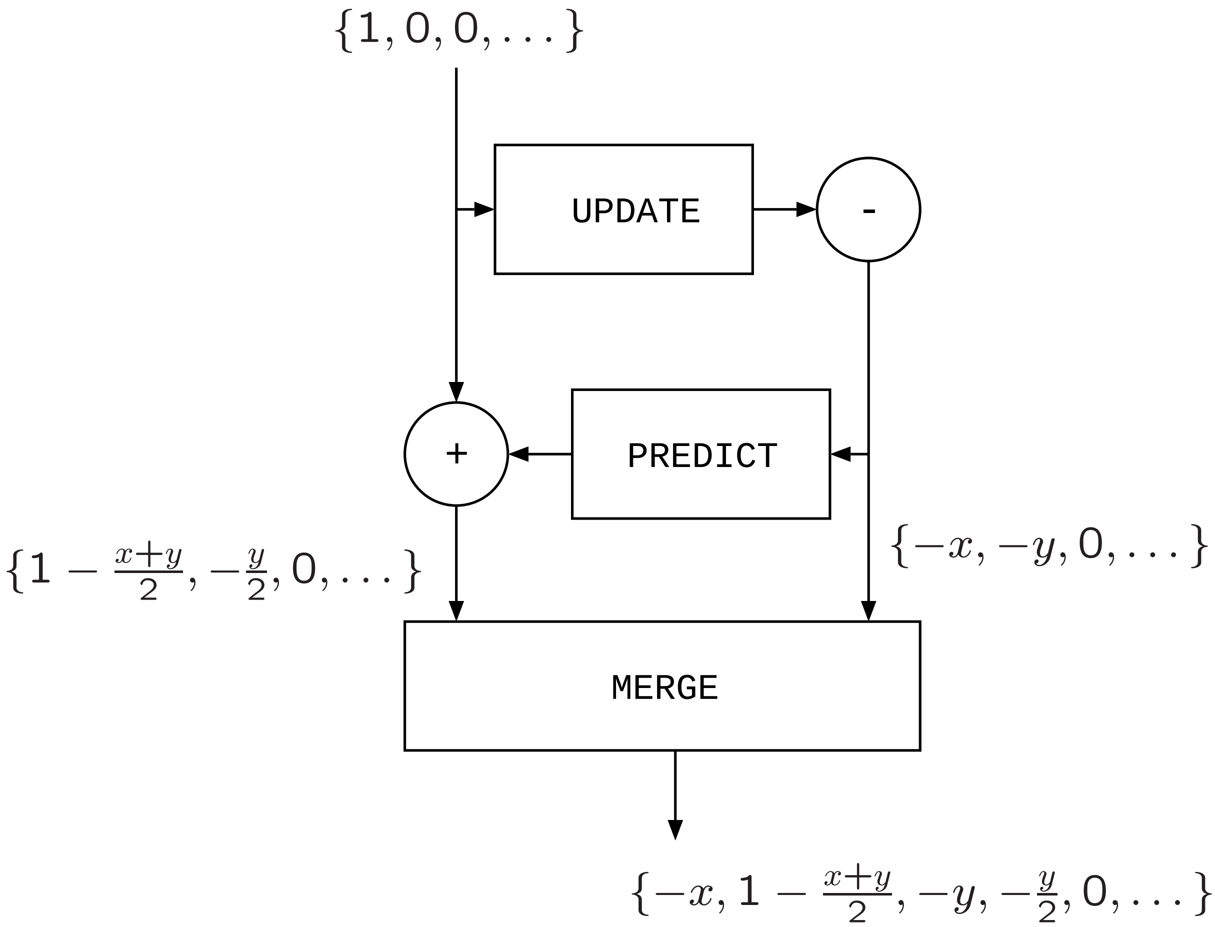




Near the boundary, we have to do something special.

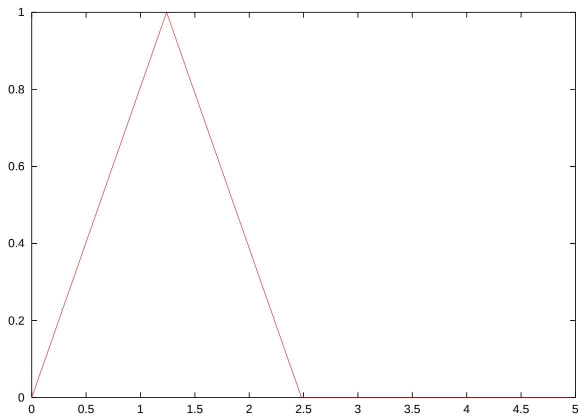
Update for interval wavelets

We design the operator U again by putting δ-pulses on the detail wire, but now we have to consider the boundary as a special case. E.g. for CDF-$(2,2)$,

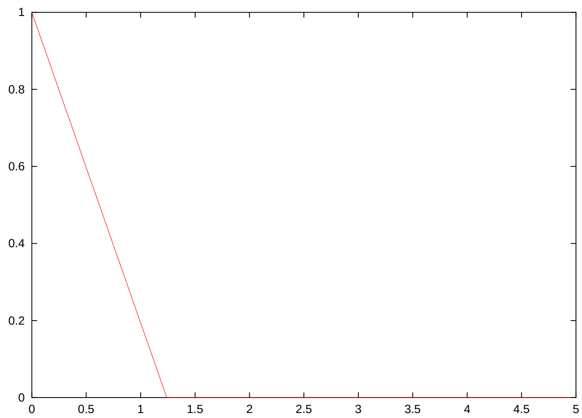


Take $x = 3/4$ and $y = 1/8$.

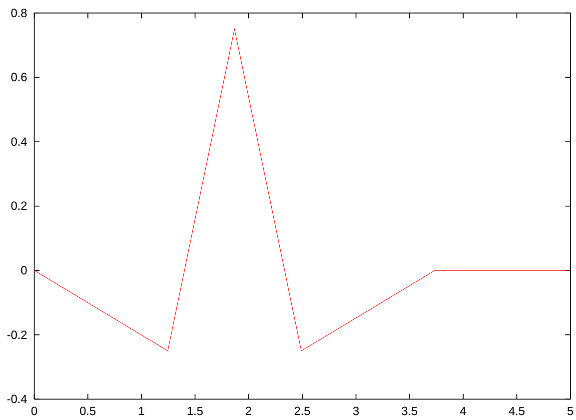
CDF-$(2,2)$ on the interval



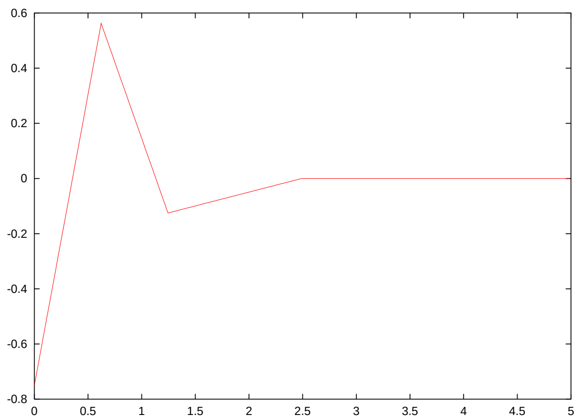
Scaling function in center



Scaling function on boundary

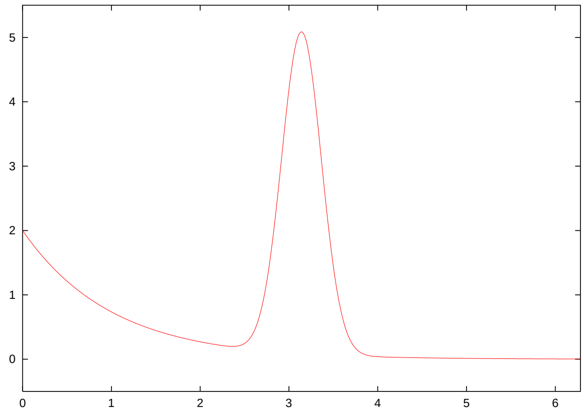


Wavelet in center

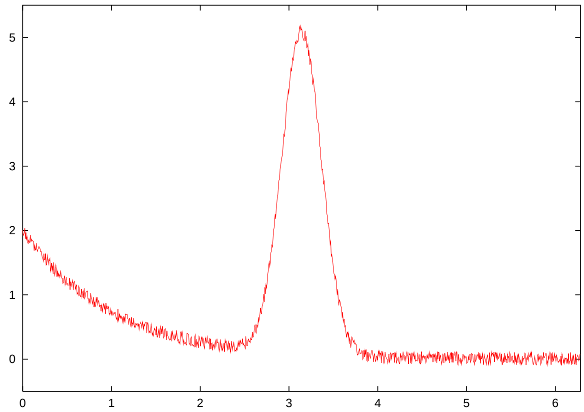


Wavelet on boundary

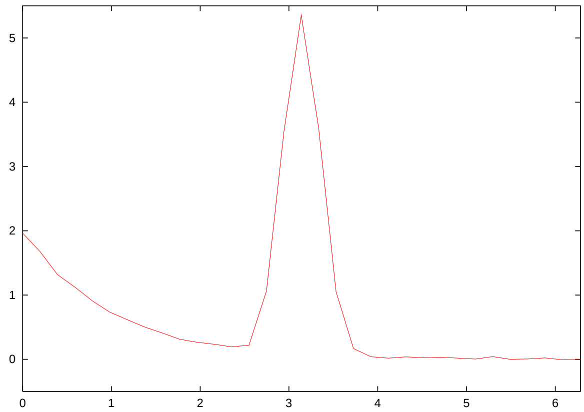
Interval CDF-$(2,2)$ for de-noising



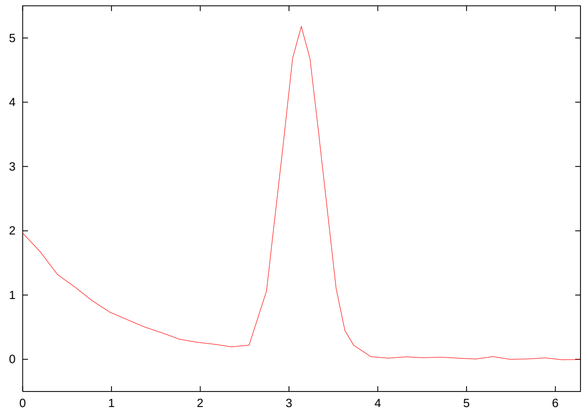
Original signal



Noisy signal



Non-adaptive filter



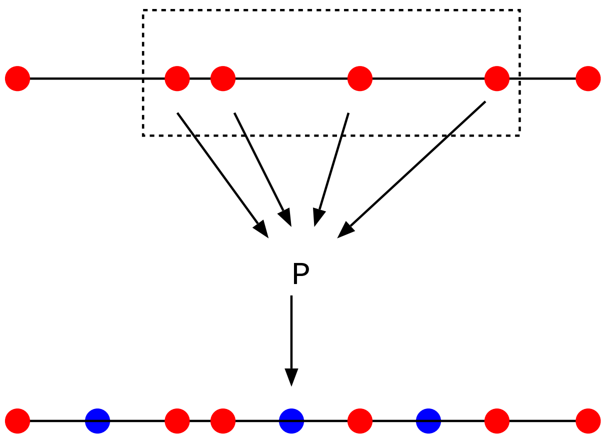
Adaptive filter

Other second-generation wavelets

1. Wavelets on non-equidistant grids
2. Wavelets with weighted inner products
3. Wavelets on a sphere (or even arbitrary surfaces)

Wavelets on non-equidistant grids

- The predict step generalises in a straightforward way, as long as we have a subdivision scheme.



- The update step is complicated by the fact that we need the averages (integrals) of the scaling functions, which may all be different. We can numerically approximate these integrals with the cascade algorithm and a quadrature rule.

Wavelets with weighted inner products

Instead of the standard L^2 inner product

$$(f,g) = \int_{-\infty}^{\infty} f(t)g(t)\,dt,$$

one may want to use a weighted inner product

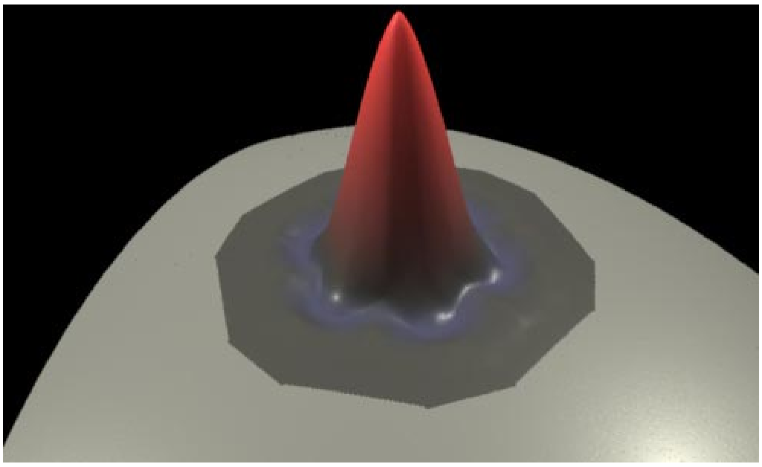
$$(f,g)_w = \int_{-\infty}^{\infty} f(t)g(t)w(t)\,dt.$$

Again, the difficulty is with the update step.
We now want the *weighted* average and the first k *weighted* moments of the wavelets to be 0.

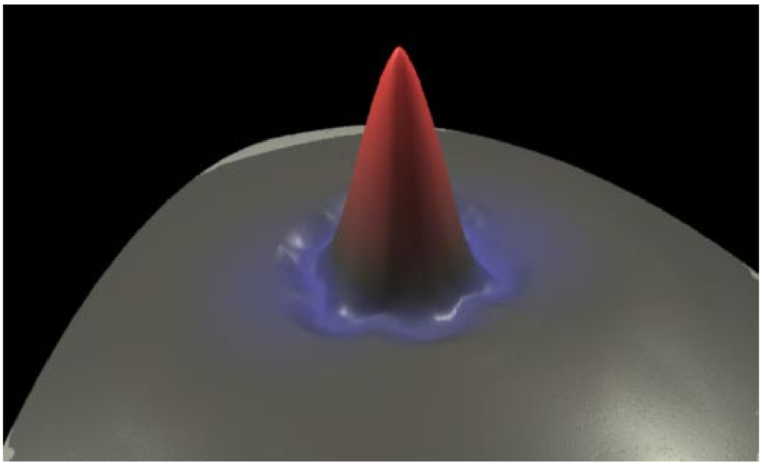
$$\int_{-\infty}^{\infty} \psi_{i,n}(t)w(t)t^p\,dt = 0, \quad p = 0, 1, \ldots, k-1.$$

The update step has to ensure all these conditions.

Wavelets on a sphere (or even arbitrary surfaces)



“Butterfly” scaling function



“Butterfly” wavelet

Images by Wim Sweldens

THE END