Purely algebraic domain decomposition method for the incompressible Navier-Stokes equation

Pawan Kumar[1]

[1]KU Leuven, Leuven, Belgium

Outline

Navier-Stokes equation

Discrete form of NS

Mini Schur complement preconditioners

Summary and future directions

Navier-Stokes equation

Consider the following time evolving Navier Stokes equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f & \qquad \text{in } \Omega \\ \nabla \cdot u = 0 & \qquad \text{in } \Omega \\ \mathtt{B} u = g & \qquad \text{on } \Gamma \end{aligned}$$

- Given:
  - Δ is Laplace op., ∇ the grad., $\nabla \cdot$ is div., and $\mathtt{B}$ is bdry operator
  - viscosity $\nu > 0$, $\nu = 1/Re$, where Re is Reynolds number
  - $\Omega \subset \mathbb{R}^d (d = 2, 3)$, bounded, connected, piece-wise smooth Γ
- To find:
  - velocity field $u : \Omega \to \mathbb{R}^d$
  - pressure field $p : \Omega \to \mathbb{R}$

Discrete form of NS

- Implicit time discretization + spatial discretization leads to

$$\mathbf{C} = \left[\begin{array}{c|c} D & E^T \\ \hline -E & 0 \end{array}\right]$$

- In 2D, we have

$$\mathbf{D} = \left[\begin{array}{c|c} D11 & 0 \\ \hline 0 & D22 \end{array}\right]$$

- D discrete form of reaction, diffusion, convection terms
- D is positive definite, $D + D^T$ is SPD
- E^T is discrete gradient, $-E$ is discrete divergence
- General problem is saddle point problem
- Other sources include constrained optimization, weighted least squares estimation

Mini Schur complement methods

- Consider the general (G not zero) 2×2 block partitioned matrix
$$\mathbf{C} = \left[\begin{array}{c|c} D & E \\ \hline F & G \end{array}\right]$$
- $S = G - FD^{-1}E$ is the global Schur complement.
- Let $\mathscr{D}$, $\mathscr{G}$ set of vertices of matrices D and G.
- Imagine graphs with vertices $\mathscr{D}$ and $\mathscr{G}$ connected by F and E
- Consider aggregation of vertices $\mathscr{G}$:
  - Non-overlapping case
    - $\mathscr{V} = \{\mathscr{G}_{p_1}, \mathscr{G}_{p_2}, \ldots, \mathscr{G}_{p_k}\}$, $\mathscr{G}_{p_i} \subset \mathscr{G}$, $\mathscr{G}_{p_i} \cap \mathscr{G}_{p_j} = \emptyset$ for $i \neq j$, $\mid \mathscr{G}_{p_i} \mid = p_i$ and $\cup_i \mathscr{G}_{p_i} = \mathscr{G}$
  - Overlapping case:
    - $\mathscr{V} = \{\mathscr{G}_{p_1}, \mathscr{G}_{p_2}, \ldots, \mathscr{G}_{p_k}\}$, $\mathscr{G}_{p_i} \subset \mathscr{G}$, $\mathscr{G}_{p_i} \cap \mathscr{G}_{p_j} \neq \emptyset$ for $i \neq j$, $\mid \mathscr{G}_{p_i} \mid = p_i$ and $\cup_i \mathscr{G}_{p_i} = \mathscr{G}$.

Construction of MSC

- Consider again

$$\mathbf{C} = \left[\begin{array}{c|c} D & E \\ \hline F & G \end{array}\right]$$

- Consider aggregates $\mathscr{V} = \{\mathscr{G}_{p_1}, \mathscr{G}_{p_2}, \ldots, \mathscr{G}_{p_k}\}$
- Identify aggregates of $\mathscr{D}$ corresp. to those of $\mathscr{G}$
  - Aggregation based on edge connectivity (MSCE)
    - For each $\mathscr{G}_{p_i}$, identify $\mathscr{D}_{r_i}$ in the set $\mathscr{D}$ such that for each node of $\mathscr{G}_{p_i}$, there exist a node in $\mathscr{D}_{r_i}$ within a path length of r_i
  - Aggregation based on direct pairing (MSCN)
    - For each $\mathscr{G}_{p_i}$, identify $\mathscr{D}_{r_i}$ in the set $\mathscr{D}$ by direct pairing
- Consider the following matrix

$$\mathbf{C}_{\mathbf{q_i},\mathbf{r_i}} = \left[\begin{array}{c|c} D_{r_i} & E_i \\ \hline F_i & G_{p_i} \end{array}\right].$$

- The i^{th} mini Schur complement is

$$S_i = G_{p_i} - F_i(D_{r_i})^{-1}E_i$$

- The i^{th} mini Schur complement: $S_i = G_{p_i} - F_i(D_{r_i})^{-1}E_i$
- The global Schur complement is $S = blkDiag(S_i)$

Let $D = \text{blkDiag}(D_1,\ D_2)$ and

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}_{n\times n} \quad E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}_{n\times n} \quad F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}_{n\times n}$$

Schur complement S is

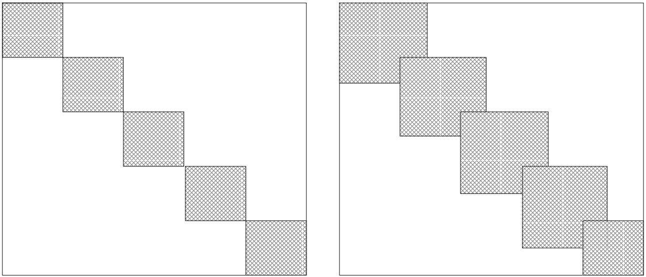
$$S = \begin{bmatrix} G_{11} - E_{11}D_1^{-1}F_{11} - E_{12}D_2^{-1}F_{21} & G_{12} - E_{11}D_1^{-1}F_{12} - E_{12}D_2^{-1}F_{22} \\ G_{21} - E_{21}D_1^{-1}F_{11} - E_{22}D_2^{-1}F_{21} & G_{22} - E_{21}D_1^{-1}F_{12} - E_{22}D_2^{-1}F_{22} \end{bmatrix}$$

MSCN approximation of the Schur complement

$$\widehat{S} = \begin{bmatrix} G_{11} - E_{11}D_1^{-1}F_{11} & \\ & G_{22} - E_{22}D_2^{-1}F_{22} \end{bmatrix}.$$

Picture overlapping and non-overlapping

- Left: structure of approximated Schur complement $\widehat{\mathbf{S}}_5$ for MSCN Right: structure of the approximated Schur complement $\widehat{\mathbf{S}}_5$ for OMSCN.



- In the overlapping case, the global Schur complement not block diagonal

Figure: An example of an aggregation. Left: A aggregated set of $\mathscr{G}$ containing two nodes indicated in bold spheres. Right: An aggregated set of $\mathscr{D}$ containing two nodes indicated in spheres with pattern.

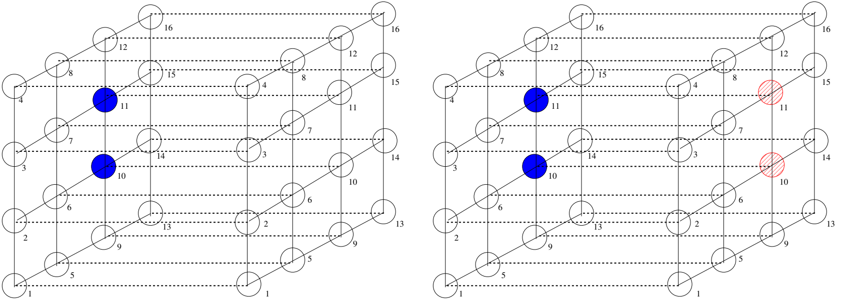


Table: Preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, stretched grids), viscosity 0.1

grid	sz	nA	nS	sA	tolA	MSCN			LUM			MSCE		
						its	time	ff	its	time	ff	its	time	ff
16	13	5	10	100	e-4	49	0.3	1.6	49	0.3	1.6	8	51	1.6
32	25	5	17	400	e-4	74	1.4	3.0	74	1.4	3.0	-	-	-
64	101	9	11	900	e-4	65	6.6	4.1	86	8.4	4.1	-	-	-

Table: ILU preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, stretched grids), viscosity 0.1

grid	ILU(e^{-1})			ILU(e^{-2})			ILU(e^{-3})			ILU(e^{-4})		
	its	time	ff	its	time	ff	its	time	ff	its	time	ff
16	30	5.9e-2	1.3	8	6.7e-2	3.8	4	8.3e-2	6.8	3	1.1e-1	9.0
32	80	6.8e-1	1.7	12	8.2e-1	5.1	6	1.6	10.5	4	2.5	16.4
64	PZ	NA	NA	31	20.6	12.4	10	40.1	23.4	5	69.5	37.1

PZ: Pivot zero, NA: not applicable

Table: Preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, stretched grids), viscosity 0.01

Grid	sz	nA	nS	sA	tolA	MSCN			LUM			MSCE		
						its	time	ff	its	time	ff	its	time	ff
16	13	5	10	100	e-4	57	0.4	1.9	57	0.4	1.9	7	51	2.0
32	25	5	17	400	e-4	80	1.5	3.3	80	1.5	3.3	-	-	-
64	101	9	11	900	e-4	64	6.4	3.8	87	7.9	3.8	-	-	-

Table: ILU preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, stretched grids), viscosity 0.01

grid	ILU(e^{-1})			ILU(e^{-2})			ILU(e^{-3})			ILU(e^{-4})		
	its	time	ff	its	time	ff	its	time	ff	its	time	ff
16	21	4.1e-2	1.0	6	5.4e-2	3.2	4	7.6e-2	5.9	3	1.0e-1	8.6
32	65	5.3e-1	1.5	11	7.8e-1	5.1	5	1.5	10.7	4	2.4	17.2
64	PZ	NA	NA	25	19.0	11.4	8	43.2	25.6	5	76.7	40.2

PZ: Pivot zero, NA: not applicable

Table: Preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, stretched grids), viscosity 0.001

Grid	sz	nA	nS	sA	tolA	MSCN			LUM			MSCE		
						its	time	ff	its	time	ff	its	time	ff
16	13	5	10	100	e-4	99	0.5	2.0	99	0.5	2.0	7	51	2.0
32	25	5	17	400	e-4	395	6.8	4.3	395	6.7	4.3	-	-	-
32	271	9	2	200	e-4	48	2.0	3.4	81	2.1	2.9	-	-	-
64	101	9	11	900	e-4	64	6.4	3.8	87	7.9	3.8	-	-	-

Table: ILU preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, stretched grids), viscosity 0.001

grid	ILU(e^{-1})			ILU(e^{-2})			ILU(e^{-3})			ILU(e^{-4})		
	its	time	ff	its	time	ff	its	time	ff	its	time	ff
16	MS	NA	NA	19	1.3e-1	6.4	5	1.3e-1	10.1	2	1.3e-1	11.7
32	$>$ 1000	NA	NA	13	1.0	6.9	5	1.8	13.4	3	2.8	20.8
64	$>$ 1000	NA	NA	19	12.4	9.0	7	27.9	19.1	4	54.5	33.8

MS: Matrix is singular

Table: Preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, uniform grids), viscosity 0.1

Grid	sz	nA	nS	sA	tolA	MSCN			LUM			MSCE		
						its	time	ff	its	time	ff	its	time	ff
16	25	5	5	100	e-4	43	0.3	1.6	43	0.3	1.6	5	44	1.8
32	38	5	11	400	e-4	67	1.3	3.2	67	1.3	3.2	-	-	-
64	69	10	16	800	e-4	111	11.2	4.6	111	11.1	4.6	-	-	-
64	138	10	8	800	e-4	63	7.2	4.6	90	7.6	4.6	-	-	-

Table: ILU preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, uniform grids), viscosity 0.1

grid	ILU(e^{-1})			ILU(e^{-2})			ILU(e^{-3})			ILU(e^{-4})		
	its	time	ff	its	time	ff	its	time	ff	its	time	ff
16	37	8.5e-2	2.4	9	9.1e-2	6.3	4	9.8e-2	8.9	3	1.1e-1	10.3
32	156	2.8	7.2	15	2.2	15.7	6	3.0	23.5	4	3.8	29.0
64	$>$ 1000	NA	NA	56	123.3	58.6	11	154.1	71.0	5	174.7	89.3

Table: Preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, uniform grids), viscosity 0.001

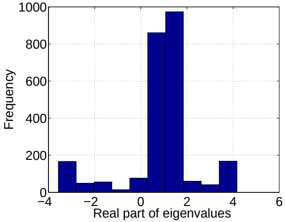
Grid	sz	nA	nS	sA	tolA	MSCN			LUM			MSCE		
						its	time	ff	its	time	ff	its	tm	ff
16	25	5	5	100	e-4	93	0.4	1.9	93	0.5	1.9	5	44	2.1
32	38	5	11	400	e-4	96	1.9	4.1	96	1.9	4.1	-	-	-
64	69	10	16	800	e-4	>1K	91	6.1	>1K	91	6.1	-	-	-
64	138	10	8	800	e-4	89	10.2	6.2	808	74.9	6.2	-	-	-

Table: ILU preconditioned GMRES on steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, uniform grids), viscosity 0.001

grid	ILU(e^{-1})			ILU(e^{-2})			ILU(e^{-3})			ILU(e^{-4})		
	its	time	ff	its	time	ff	its	time	ff	its	time	ff
16	400	1.5	3.1	22	1.8e-1	8.8	6	1.5e-1	11.4	3	1.5e-1	12.11
32	MS	NA	NA	13	1.4	10.3	5	2.4	18.1	3	3.5	26.2
64	>1000	NA	NA	23	28.3	20.8	6	59.8	38.2	3	96.7	56.9

MS: Matrix is singular

Table: Top Left: Real part of the eigenvalues of original matrix, Top Right Real part of the eigenvalues of the MSCN right preconditioned matrix



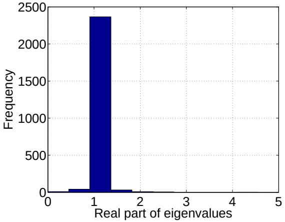
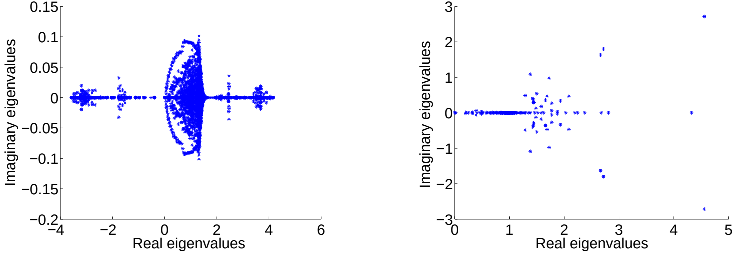


Table: Top Left: Spectrum of of original matrix, Top Right Spectrum of the MSCN right preconditioned matrix



Theorem on eigenvalues

- Consider the block 2×2 matrix

$$\mathbf{C} = \begin{bmatrix} D_{p\times p} & E \\ F & G \end{bmatrix}_{n\times n},$$

- Let $\mathbf{B_{MSCN}}$ be the MSCN preconditioner.
- Conclusion
  - The MSCN preconditioned matrix $\mathbf{C}$ i.e., $(\mathbf{B_{MSCN}})^{-1}\mathbf{C}$ has p eigenvalues exactly equal one
  - The rest of the n-p eigenvalues are the eigenvalues of $(\widehat{S})^{-1}S$
  - where $S = G - FD^{-1}E$ is the global Schur complement of the matrix $\mathbf{C}$.
- For proof see the article uploaded on arXiv

Table: maxits: 3000, GMRES(300), steady Oseen problems, leaky lid driven cavity, (Q2-Q1 FEM, uniform grids), for OMSCN, MPCD, and LSC

Re	$\frac{1}{h}$	sz	nA	nS	sA	OMSCN			MPCD		LSC	
						its	tm	ff	its	tm	its	tm
10	32	38	5	11	400	52	0.7	3.1	24	1.7	13	0.9
	64	69	10	16	800	57	4.8	4.6	25	7.8	17	4.5
	128	107	12	31	2700	104	50	7.0	25	31	21	26
100	32	38	5	11	400	53	0.7	3.5	46	2.6	31	2.2
	64	69	10	16	800	61	4.2	4.1	48	12.5	35	9.1
	128	107	12	31	2700	118	46	6.2	49	55	49	56
500	32	38	5	11	400	66	10.4	4.2	120	8.8	96	5.5
	64	69	10	16	800	64	5.7	5.9	117	28.4	101	26.9
	128	107	12	31	2700	108	58	8.8	108	119	100	121
1000	32	38	5	11	400	83	1.2	4.2	179	10.5	155	9.0
	64	69	10	16	800	82	7.4	6.2	213	53.2	191	52.1
	128	107	12	31	2700	104	49	7.0	198	225	187	231
3000	32	38	5	11	400	232	4.6	4.8	302	18.5	300	18.6
	64	69	10	16	800	183	19	7.0	1127	362.3	1103	381.8
	128	107	12	31	2700	854	705	11.4	NC	NC	NC	NC

- LSC stands for Least square commutator
- MPCD stands for Modified pressure convection-diffusion

Summary

- Introduced overlapping and non-overlapping versions of mini-Schur complement method
- The proposed method is parallel during both construction and solve process
- For low viscosity value, the iteration count and CPU time for the proposed methods are less than that of MPCD and LSC methods
- Future work consists of parallel implementation of the proposed methods

For Further Reading I

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For Further Reading II

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THANK YOU!