Micro-/macro-block factorizations for regularized saddle point systems

Joseph M.L. Maubach and Wil H.A. Schilders

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Introduction

Any symmetric matrix $\mathbf{X}$, positive definite or not, can be factored:

$$\mathbf{Q}^{\mathrm{T}}\mathbf{X}\mathbf{Q} = \mathbf{L}\mathbf{D}\mathbf{L}^{\mathrm{T}}$$

where

1. $\mathbf{L}$ is a unit lower triangular matrix, and

See [3, Section 4.4, page 115].

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Regularized saddle-point problems: Definition

Structure of indefinite $\mathbf{X}$ from regularized saddle point problems:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^{\mathrm{T}} \\ \mathbf{B} & -\mathbf{C} \end{bmatrix}.$$

1. $\mathbf{A}$ n by n symmetric positive definite;

Goal: Construct a preconditioner or implicit/explicit factorization.

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2. $\mathbf{C} \neq \mathbf{0}$ (regularized saddle point problem): occurs for instance in electronic circuit simulation [11].

Goal: Construct a preconditioner or implicit/explicit factorization.

Regularized saddle-point problems: Preconditioners

Are for instance based on:

1. (Block-) diagonal approximations of $\mathbf{A}$;
2. (Block-) diagonal approximations of $\mathbf{C}$;
3. Schur-complement based approaches;
4. Explicit/implicit Micro-block factorizations.

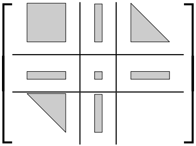
See for instance Axelsson, Gould, Keller, Schilders, Simoncini, Trefethen, Wathen [1, 5, 6, 12, 4, 2, 8]. Example approach from [5]: Factor $\mathbf{C} = \mathbf{L}\mathbf{D}\mathbf{L}^{\mathrm{T}}$ and introduce $\mathbf{x}_3 = -\mathbf{D}\mathbf{L}^{\mathrm{T}}\mathbf{x}_2$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{B}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{D}^{-1} & \mathbf{L}^{\mathrm{T}} \\ \mathbf{B} & \mathbf{L} & \mathbf{0} \end{bmatrix}.$$

Schilders' factorization/preconditioner for $\mathbf{C} = \mathbf{0}$

Schilder's in [12] exploits more structure:

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^{\mathrm{T}} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} =$$



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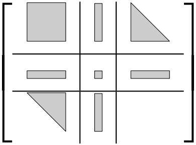

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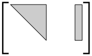
2. $\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2] =$  m by n upper trgl., max. row rank;

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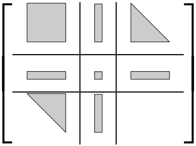

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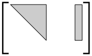
2. $\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2] =$  m by n upper trgl., max. row rank;
3. $\mathbf{C} = \mathbf{0}$, m by m – diagonal matrix.

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Schilders' I: The micro-block matrix $\mathbf{Y} = \mathbf{Q}^{\mathrm{T}}\mathbf{X}\mathbf{Q}$

[12] proposes the pivot-explicit micro-block matrix $\mathbf{Y} = \mathbf{Q}^{\mathrm{T}}\mathbf{X}\mathbf{Q}$

	1		2				m		$m+1$		n
1	a_{11}	b_{11}	a_{12}	0	a_{13}	0	a_{14}	0	a_{15}	a_{16}	a_{17}
	b_{11}	$-c_{11}$	b_{12}	0	b_{13}	0	b_{41}	0	b_{15}	b_{16}	b_{17}
	a_{21}	b_{12}	a_{22}	b_{22}	a_{23}	0	a_{24}	0	a_{25}	a_{26}	a_{27}
	0	0	b_{22}	$-c_{22}$	b_{23}	0	b_{24}	0	b_{25}	b_{26}	b_{27}
	a_{31}	b_{13}	a_{32}	b_{23}	a_{33}	b_{33}	a_{34}	0	a_{35}	a_{36}	a_{37}
	0	0	0	0	b_{33}	$-c_{33}$	b_{34}	0	b_{35}	b_{36}	b_{37}
m	a_{41}	b_{14}	a_{42}	b_{24}	a_{43}	b_{34}	a_{44}	b_{44}	a_{45}	a_{46}	a_{47}
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$m+1$	a_{51}	b_{15}	a_{52}	b_{25}	a_{53}	b_{35}	a_{54}	b_{45}	a_{55}	a_{56}	a_{57}
	a_{61}	b_{16}	a_{62}	b_{26}	a_{63}	b_{36}	a_{64}	b_{46}	a_{65}	a_{66}	a_{67}
n	a_{71}	b_{17}	a_{72}	b_{27}	a_{73}	b_{37}	a_{74}	b_{47}	a_{75}	a_{76}	a_{77}

and uses it to construct a preconditioner $\hat{\mathbf{Y}}$

1. of Crout-type: $\hat{\mathbf{Y}} = \hat{\mathcal{L}}\mathrm{diag}^{-1}(\hat{\mathcal{L}})\hat{\mathcal{L}}^{\mathrm{T}}$;

Concept 1: New factorization [9] uses this micro-block structure.

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2. which satisfies $\mathrm{diag}(\hat{\mathbf{Y}}) = \mathrm{diag}(\mathbf{Y})$.

Concept 1: New factorization [9] uses this micro-block structure.

Schilders' II: The macro-block factorization $\mathbf{X} = \mathcal{L}\mathcal{D}\mathcal{L}^{\mathrm{T}}$

In addition [12] shows that for $\mathbf{C} = \mathbf{0}$:

1. macro-block factorizations $\mathbf{X} = \mathcal{L}\mathcal{D}\mathcal{L}^{\mathrm{T}}$ exist where

$$\mathcal{L} = \begin{bmatrix} \mathbf{B}_1^{\mathrm{T}} & \mathbf{0} & \mathbf{L}_1 \\ \mathbf{B}_2^{\mathrm{T}} & \mathbf{I}_{n-m} + \mathbf{L}_2 & \mathbf{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{D}_2 & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix};$$

2. $x \mapsto \mathcal{L}^{-1}\mathbf{x}$ and $x \mapsto \mathcal{D}^{-1}\mathbf{x}$ can be calculated efficiently;
3. This factorization is not unique;
4. A full micro-block factorization leads to a different factorization unless $\mathrm{diag}(\mathbf{B}) = \mathbf{I}$.

Concept 2: New factorization [9] shows micro-block factorization existence using a different macro-block factorization.

The new factorization for $\mathbf{C} \neq \mathbf{0}$, structure of $\mathbf{X}$

The current work [9] focuses on

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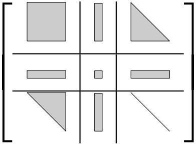

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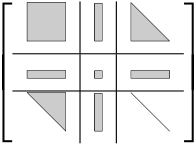

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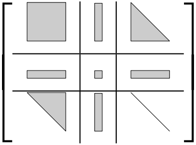

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The factorizations $\mathbf{X}=\mathbf{L_X}\mathbf{D_X}^{-1}\mathbf{L_X}^{\mathrm{T}}$, $\mathbf{X}=\mathcal{L}_\mathbf{X}\mathbf{D_X}\mathcal{L}_\mathbf{X}^{\mathrm{T}}$

For $\mathbf{C}\neq\mathbf{0}$ [9] introduces an *exact micro-block factorization*:

1. efficient, since updates are only on the **A**-block part of **X**;
2. induces factorization ($\mathbf{D_X}=\mathrm{diag}(\mathbf{L_X})$): $\mathbf{X}=\mathbf{L_X}\mathbf{D_X}^{-1}\mathbf{L_X}^{\mathrm{T}}$ where

$$\mathbf{L_X}=\begin{bmatrix}\mathbf{L}_1 & \mathbf{0} & \mathbf{B}_1^{\mathrm{T}}\\ \mathbf{M} & \mathbf{L}_2 & \mathbf{B}_2^{\mathrm{T}}\\ \mathrm{d}\mathbf{B}_1 & \mathbf{0} & -\mathbf{C}\end{bmatrix},\quad \mathbf{D_X}^{-1}=\begin{bmatrix}\mathbf{FC} & \mathbf{0} & \mathbf{F}\mathrm{d}\mathbf{B}_1\\ \mathbf{0} & \mathbf{D}_2^{-1} & \mathbf{0}\\ \mathbf{F}\mathrm{d}\mathbf{B}_1 & \mathbf{0} & -\mathbf{FD}_1\end{bmatrix};$$

3. alternative formulation $\mathbf{X}=\mathcal{L}_\mathbf{X}\mathbf{D_X}\mathcal{L}_\mathbf{X}^{\mathrm{T}}$ where

$$\begin{bmatrix}\mathbf{B}_1^{\mathrm{T}}\mathbf{F}\mathrm{d}\mathbf{B}_1+\mathbf{L}_1\mathbf{FC} & \mathbf{0} & -\mathbf{B}_1^{\mathrm{T}}\mathbf{FD}_1+\mathbf{L}_1\mathbf{F}\mathrm{d}\mathbf{B}_1\\ \mathbf{B}_2^{\mathrm{T}}\mathbf{F}\mathrm{d}\mathbf{B}_1+\mathbf{MFC} & \mathbf{L}_2\mathbf{D}_2^{-1} & -\mathbf{B}_2^{\mathrm{T}}\mathbf{FD}_1+\mathbf{MF}\mathrm{d}\mathbf{B}_1\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\end{bmatrix};$$

4. factorizations are unique (m-diag. blocks are 1x1 or 2x2 identity).

The structure of micro-block factor $\mathbf{Q}^{\mathrm{T}}\mathcal{L}_{\mathbf{X}}\mathbf{Q}$

$\mathbf{Q}^{\mathrm{T}}\mathcal{L}_{\mathbf{X}}\mathbf{Q}$ only differs from the lower triangular micro-block part of $\mathbf{Y}$ at entries a_{ij}:

	1		2				m		$m+1$		n
1	$*_{11}$	b_{11}									
	b_{11}	$-c_{11}$									
	$*_{21}$	b_{12}	$*_{22}$	b_{22}							
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Similarities between macro-factorizations in [12] and [9]

For $\mathbf{C} = \mathbf{0}$:

1. Old and new factorizations are identical for $\mathrm{diag}(\mathbf{B}_1) = \mathbf{I}_m$.

For $\mathbf{C} \geq \mathbf{0}$:

1. Blocks of Schilders' $\mathcal{L}$ and new factorization $\mathcal{L}_{\mathbf{X}}$ have identical shapes – zero, diagonal, (strictly) triangular, rectangular;
2. $\mathbf{D}_{\mathbf{X}}$ has one extra non-zero block more than $\mathcal{D}$.

The new factorization conceptually generalizes Schilders' factorization.

Existence of factorizations $\mathbf{X} = \mathbf{L}_\mathbf{X}\mathbf{D}_\mathbf{X}^{-1}\mathbf{L}_\mathbf{X}^{\mathrm{T}}$, $\mathbf{X} = \mathcal{L}_\mathbf{X}\mathbf{D}_\mathbf{X}\mathcal{L}_\mathbf{X}^{\mathrm{T}}$

Shown is existence for **A** spd., **B** upper trngl. of max. row rank

1. $\mathbf{C} = \mathbf{0}$, uses Lemma 4.1 in Schilders [12] – but differs, uses $\mathbf{B} = [\mathbf{I}_m, \mathbf{0}_{m,n-m}]$ plus transformation;
2. $\mathbf{C} > \mathbf{0}$, uses existence of Schur-complement $\mathbf{A} + \mathbf{B}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{B}$;
3. $\mathbf{C} = \mathrm{diag}(0, \ldots, 0, c_{d+1,d+1}, \ldots, c_{mm})$ with d zeros at start.

Extensions are likely and under investigation:

1. $\mathbf{C} \geq \mathbf{0}$, no order in zero and positive coefficients;
2. k by k micro-block approach (calculation of $\mathbf{D}_\mathbf{X}^{-1}$ is similar);
3. construction of preconditioners: Incomplete, etc.

General positive (semi-)definite $\mathbf{C}$ and full-rank $\mathbf{B}$

For the general case existence can be shown as follows:

1. Factor $\mathbf{C} = \mathbf{L_C D_C L_C}$ – use Cholesky form;
2. Apply a QR-based transformation to make $\hat{\mathbf{B}} = \mathbf{B}(\mathbf{L_C}\sqrt{\mathbf{D_C}})^{-1}$ upper triangular;
3. Apply the micro-block factorization;
4. Read off the macro-block factors.

The extra work is the factorization of $\mathbf{C}$.

Conclusions/Future work

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