

# Implementation and numerical stability of saddle point solvers

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Recent developments in the solution of indefinite systems

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## Saddle point problems

We consider a saddle point problem with the symmetric  $2 \times 2$  block form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

- ▶  $A$  is a square  $n \times n$  nonsingular (symmetric positive definite) matrix,
- ▶  $B$  is a rectangular  $n \times m$  matrix of (full column) rank  $m$ .

Applications: mixed finite element approximations, weighted least squares, constrained optimization, computational fluid dynamics, electromagnetism etc. [Benzi, Golub and Liesen, 2005], [Elman, Silvester, Wathen, 2005]. For the updated list of applications leading to saddle point problems contact [Benzi].

PROBLEM

SOLUTION APPROACH

PRECONDITIONER

ITERATIVE SOLVER

## Iterative solution of saddle point problems

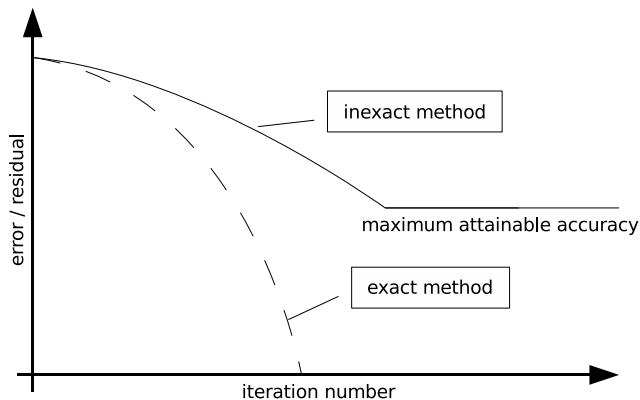
1. **segregated approach**: outer iteration for solving the reduced Schur complement or null-space projected system;
2. **coupled approach with block preconditioning**: iteration scheme for solving the preconditioned system;
3. **rounding errors in floating point arithmetic**: numerical stability of the solver

Numerous solution schemes: inexact Uzawa algorithms, inexact null-space methods, inner-outer iteration methods, two-stage iteration processes, multilevel or multigrid methods, domain decomposition methods

Numerous preconditioning techniques and schemes: block diagonal preconditioners, block triangular preconditioners, constraint preconditioning, Hermitian/skew-Hermitian preconditioning and other splittings, combination preconditioning

Numerous iterative solvers: conjugate gradient (CG) method, MINRES, GMRES, flexible GMRES, GCR, BiCG, BiCGSTAB, ...

## Delay of convergence and limit on the final accuracy



## Numerical experiments: a small model example

$A = \text{tridiag}(1, 4, 1) \in \mathbb{R}^{100 \times 100}$ ,  $B = \text{rand}(100, 20)$ ,  $f = \text{rand}(100, 1)$ ,

$$\kappa(A) = \|A\| \cdot \|A^{-1}\| = 5.9990 \cdot 0.4998 \approx 2.9983,$$

$$\kappa(B) = \|B\| \cdot \|B^\dagger\| = 7.1695 \cdot 0.4603 \approx 3.3001.$$

## Schur complement reduction method

- ▶ Compute  $y$  as a solution of the Schur complement system

$$B^T A^{-1} B y = B^T A^{-1} f,$$

- ▶ compute  $x$  as a solution of

$$A x = f - B y.$$

- ▶ Segregated vs. coupled approach:  $x_k$  and  $y_k$  approximate solutions to  $x$  and  $y$ , respectively.
- ▶ Inexact solution of systems with  $A$ : **every computed solution  $\hat{u}$  of  $Au = b$  is interpreted as an exact solution of a perturbed system**

$$(A + \Delta A)\hat{u} = b + \Delta b, \quad \|\Delta A\| \leq \tau \|A\|, \quad \|\Delta b\| \leq \tau \|b\|, \quad \tau \kappa(A) \ll 1.$$

## Iterative solution of the Schur complement system

choose  $y_0$ , solve  $Ax_0 = f - By_0$

compute  $\alpha_k$  and  $p_k^{(y)}$

$$y_{k+1} = y_k + \alpha_k p_k^{(y)}$$

$$\left. \begin{array}{l} \text{solve } Ap_k^{(x)} = -Bp_k^{(y)} \end{array} \right\}$$

**back-substitution:**

$$\mathbf{A}: x_{k+1} = x_k + \alpha_k p_k^{(x)},$$

$$\mathbf{B}: \text{solve } Ax_{k+1} = f - By_{k+1},$$

$$\mathbf{C}: \text{solve } Au_k = f - Ax_k - By_{k+1},$$

$$x_{k+1} = x_k + u_k.$$

inner  
iteration

outer  
iteration

$$r_{k+1}^{(y)} = r_k^{(y)} - \alpha_k B^T p_k^{(x)}$$



## Accuracy in the saddle point system

$$\|f - Ax_k - By_k\| \leq \frac{O(\alpha_1)\kappa(A)}{1 - \tau\kappa(A)} (\|f\| + \|B\|Y_k),$$

$$\| -B^T x_k - r_k^{(y)} \| \leq \frac{O(\alpha_2)\kappa(A)}{1 - \tau\kappa(A)} \|A^{-1}\| \|B\| (\|f\| + \|B\|Y_k),$$

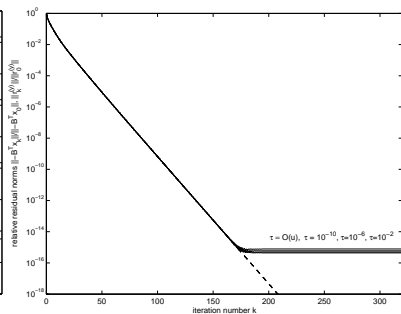
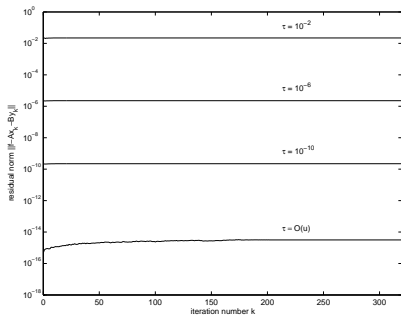
$$Y_k \equiv \max\{\|y_i\| \mid i = 0, 1, \dots, k\}.$$

Back-substitution scheme	$\alpha_1$	$\alpha_2$
<b>A:</b> Generic update $x_{k+1} = x_k + \alpha_k p_k^{(x)}$	$\tau$	$u$
<b>B:</b> Direct substitution $x_{k+1} = A^{-1}(f - By_{k+1})$	$\tau$	$\tau$
<b>C:</b> Corrected dir. subst. $x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1})$	$u$	$\tau$

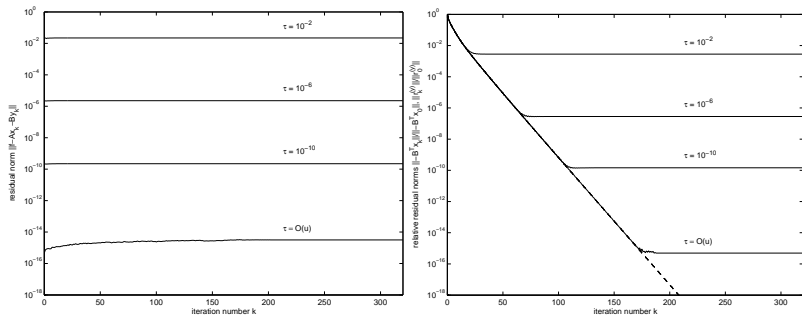
} additional system with A

$$-B^T A^{-1} f + B^T A^{-1} B y_k = -B^T x_k - B^T A^{-1} (f - Ax_k - B y_k)$$

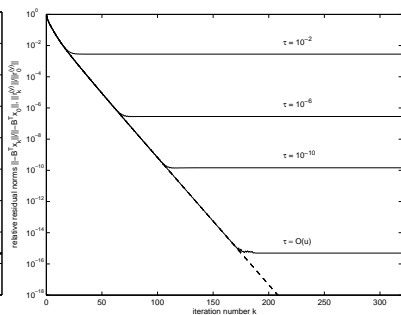
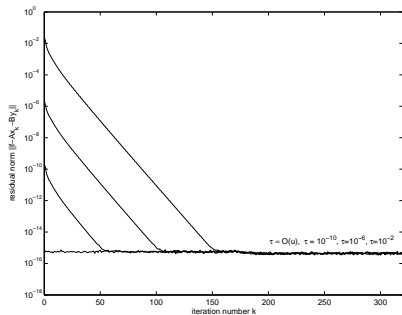
Generic update:  $x_{k+1} = x_k + \alpha_k p_k(x)$



Direct substitution:  $x_{k+1} = A^{-1}(f - By_{k+1})$



Corrected direct substitution:  $x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1})$



## Null-space projection method

- ▶ compute  $x \in N(B^T)$  as a solution of the projected system

$$(I - \Pi)A(I - \Pi)x = (I - \Pi)f,$$

- ▶ compute  $y$  as a solution of the least squares problem

$$By \approx f - Ax,$$

$\Pi = B(B^T B)^{-1} B^T$  is the orthogonal projector onto  $R(B)$ .

- ▶ Schemes with the inexact solution of least squares with  $B$ . Every computed approximate solution  $\bar{v}$  of a least squares problem  $Bv \approx c$  is interpreted as an exact solution of a perturbed least squares

$$(B + \Delta B)\bar{v} \approx c + \Delta c, \quad \|\Delta B\| \leq \tau \|B\|, \quad \|\Delta c\| \leq \tau \|c\|, \quad \tau \kappa(B) \ll 1.$$

## Null-space projection method

choose  $x_0$ , solve  $By_0 \approx f - Ax_0$

compute  $\alpha_k$  and  $p_k^{(x)} \in N(B^T)$

$$x_{k+1} = x_k + \alpha_k p_k^{(x)}$$

solve  $Bp_k^{(y)} \approx r_k^{(x)} - \alpha_k Ap_k^{(x)}$

**back-substitution:**

**A:**  $y_{k+1} = y_k + p_k^{(y)}$ ,

**B:** solve  $By_{k+1} \approx f - Ax_{k+1}$ ,

**C:** solve  $Bv_k \approx f - Ax_{k+1} - By_k$ ,

$$y_{k+1} = y_k + v_k.$$

$$r_{k+1}^{(x)} = r_k^{(x)} - \alpha_k Ap_k^{(x)} - Bp_k^{(y)}$$

} inner  
iteration

} outer  
iteration

## Accuracy in the saddle point system

$$\|f - Ax_k - By_k - r_k^{(x)}\| \leq \frac{O(\alpha_3)\kappa(B)}{1 - \tau\kappa(B)} (\|f\| + \|A\|X_k),$$

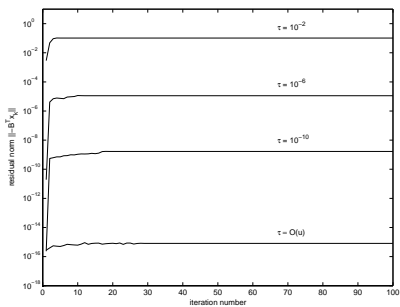
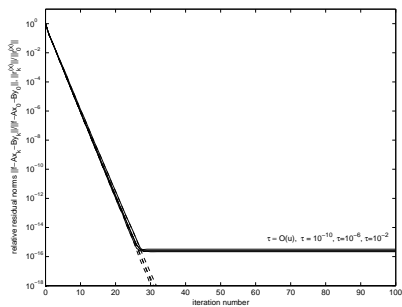
$$\| -B^T x_k \| \leq \frac{O(\tau)\kappa(B)}{1 - \tau\kappa(B)} \|B\|X_k,$$

$$X_k \equiv \max\{\|x_i\| \mid i = 0, 1, \dots, k\}.$$

Back-substitution scheme	$\alpha_3$
<b>A:</b> Generic update $y_{k+1} = y_k + p_k^{(y)}$	$u$
<b>B:</b> Direct substitution $y_{k+1} = B^\dagger(f - Ax_{k+1})$	$\tau$
<b>C:</b> Corrected dir. subst. $y_{k+1} = y_k + B^\dagger(f - Ax_{k+1} - By_k)$	$u$

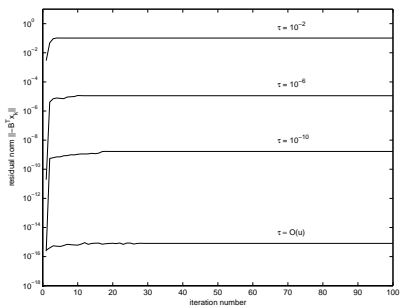
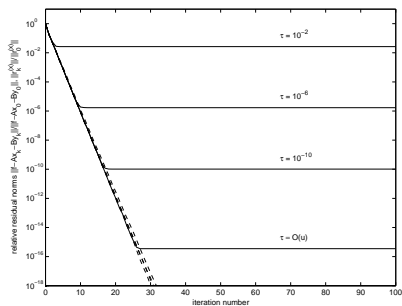
} additional least square with B

Generic update:  $y_{k+1} = y_k + p_k^{(y)}$

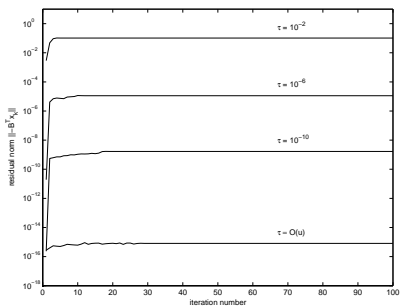
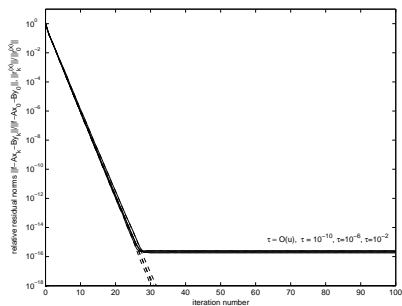




Direct substitution:  $y_{k+1} = B^\dagger(f - Ax_{k+1})$



Corrected direct substitution:  $y_{k+1} = y_k + B^\dagger(f - Ax_{k+1} - By_k)$



## Stationary iterative methods

▶  $\mathcal{A} = \mathcal{A}x = b, \mathcal{M} - \mathcal{N}$

▶ A:  $\mathcal{M}x_{k+1} = \mathcal{N}x_k + b$

B:  $x_{k+1} = x_k + \mathcal{M}^{-1}(b - \mathcal{A}x_k)$

- ▶ Inexact solution of systems with  $\mathcal{M}$ : **every computed solution  $\bar{y}$  of  $\mathcal{M}y = z$  is interpreted as an exact solution of a perturbed system**

$$(\mathcal{M} + \Delta\mathcal{M})\bar{y} = z, \quad \|\Delta\mathcal{M}\| \leq \tau\|\mathcal{M}\|, \quad \tau k(\mathcal{M}) \ll 1$$

## Accuracy of the computed approximate solution

A  $\mathcal{M}x_{k+1} = \mathcal{M}x_k + b$

$$\frac{\|\hat{x}_{k+1} - x\|_\infty}{\|x\|_\infty} \leq \tau \frac{\|\mathcal{M}^{-1}\| \|\mathcal{M}\| \|x\|_\infty}{\|x\|_\infty}$$

B  $x_{k+1} = x_k + \mathcal{M}^{-1}(b - \mathcal{A}x_k)$

$$\frac{\|\hat{x}_{k+1} - x\|_\infty}{\|x\|_\infty} \leq O(u) \frac{\|\mathcal{M}^{-1}\| \|\mathcal{A}\| \|x\|_\infty}{\|x\|_\infty}$$

new\_value = old\_value + small\_correction

## Two-stage iterative methods

$$\mathcal{M}_1 x_{k+1/2} = \mathcal{N}_1 x_k + b, \quad \mathcal{A} = \mathcal{M}_1 - \mathcal{N}_1$$

$$\mathcal{M}_2 x_{k+1} = \mathcal{N}_2 x_{k+1/2} + b, \quad \mathcal{A} = \mathcal{M}_2 - \mathcal{N}_2$$

$$x_{k+1/2} = x_k + \mathcal{M}_1^{-1}(b - \mathcal{A}x_k)$$

$$x_{k+1} = x_{k+1/2} + \mathcal{M}_2^{-1}(b - \mathcal{A}x_{k+1/2})$$

$\Leftrightarrow$

$$x_{k+1} = x_k + (\mathcal{M}_1^{-1} + \mathcal{M}_2^{-1} - \mathcal{M}_2^{-1}\mathcal{A}\mathcal{M}_1^{-1})(b - \mathcal{A}x_k)$$

$$= x_k + (\mathcal{I} + \mathcal{M}_2^{-1}\mathcal{N}_1)\mathcal{M}_1^{-1}(b - \mathcal{A}x_k)$$

$$= x_k + \mathcal{M}_2^{-1}(\mathcal{I} + \mathcal{N}_2\mathcal{M}_1^{-1})(b - \mathcal{A}x_k)$$

$$\frac{\|\hat{x}_{k+1} - x\|_\infty}{\|x\|_\infty} \leq O(u) \frac{\|\mathcal{M}_2^{-1}(\mathcal{I} + \mathcal{N}_2\mathcal{M}_1^{-1})\| \|\mathcal{A}\| \|x\|_\infty}{\|x\|_\infty}$$

$\mathcal{A}$  symmetric indefinite,  $\mathcal{P}$  positive definite

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \approx \mathcal{P} = \mathcal{R}^T \mathcal{R}$$

$$(\mathcal{R}^{-T} \mathcal{A} \mathcal{R}^{-1}) \mathcal{R} \begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{R}^{-T} \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$\mathcal{R}^{-T} \mathcal{A} \mathcal{R}^{-1}$  is symmetric indefinite!

$\mathcal{P}$  symmetric indefinite or nonsymmetric

$$\mathcal{P}^{-1} \mathcal{A} \begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{P}^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$$(\mathcal{A}\mathcal{P}^{-1}) \mathcal{P} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$\mathcal{P}^{-1} \mathcal{A}$  and  $\mathcal{A}\mathcal{P}^{-1}$  are nonsymmetric!

## Schur complement approach with indefinite preconditioner

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} A & B \\ B^T & B^T A^{-1} B - I \end{pmatrix}$$

$$\mathcal{A}\mathcal{P}^{-1} = \begin{pmatrix} I & 0 \\ (I - S)B^T A^{-1} & S \end{pmatrix}$$

$S = B^T A^{-1} B$ ,  $\mathcal{A}\mathcal{P}^{-1}$  **nonsymmetric** but **diagonalizable** and it has a 'nice' spectrum!

$$\sigma(\mathcal{A}\mathcal{P}^{-1}) \subset \{1\} \cup \sigma(B^T A^{-1} B^T)$$

[Durazzi, Ruggiero 2003], [Fortin, El-Maliki, 2009?]



## Krylov method with the preconditioner: basic properties

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, r_0 = \begin{pmatrix} 0 \\ s_0 \end{pmatrix}, e_{k+1} = \begin{pmatrix} x - x_{k+1} \\ y - y_{k+1} \end{pmatrix}$$

$$r_{k+1} = \begin{pmatrix} f \\ 0 \end{pmatrix} - \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix}$$

$$\begin{aligned} r_0 = \begin{pmatrix} 0 \\ s_0 \end{pmatrix} &\Rightarrow r_{k+1} = \begin{pmatrix} 0 \\ s_{k+1} \end{pmatrix} \\ &\Rightarrow Ax_{k+1} + By_{k+1} = f \end{aligned}$$

## Preconditioned CG method: saddle point problem and indefinite preconditioner

$$r_{k+1}^T \mathcal{P}^{-1} r_j = 0, \quad j = 0, \dots, k$$

$y_{k+1}$  is an iterate from CG applied to the Schur complement system

$$B^T A^{-1} B y = B^T A^{-1} f!$$

satisfying

$$\min_{u \in x_0 + K_{k+1}(B^T A^{-1} B, B^T A^{-1} f)} \|y - u\|_{B^T A^{-1} B} =$$

## Preconditioned CG algorithm

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, r_0 = b - \mathcal{A} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ s_0 \end{pmatrix}$$

$$\begin{pmatrix} p_0^{(x)} \\ p_0^{(y)} \end{pmatrix} = \mathcal{P}^{-1} r_0 = \mathcal{P}^{-1} \begin{pmatrix} 0 \\ s_0 \end{pmatrix}$$

$k = 0, 1, \dots$

$$\alpha_k = \left( \begin{pmatrix} 0 \\ s_k \end{pmatrix}, \mathcal{P}^{-1} \begin{pmatrix} 0 \\ s_k \end{pmatrix} \right) / \left( \mathcal{A} \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix}, \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix} \right)$$

$$\alpha_k = \frac{(r_k, z_k)}{(\mathcal{A}p_k, p_k)}$$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \alpha_k \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix}$$

$$r_{k+1} = r_k - \alpha_k \mathcal{A} \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix} = \begin{pmatrix} 0 \\ s_{k+1} \end{pmatrix}$$

$$z_{k+1} = \mathcal{P}^{-1} r_{k+1}$$

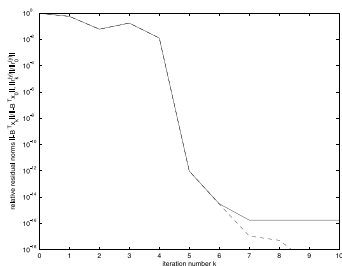
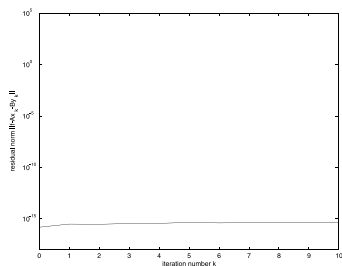
$$\beta_k = \left( \begin{pmatrix} 0 \\ s_{k+1} \end{pmatrix}, \mathcal{P}^{-1} \begin{pmatrix} 0 \\ s_{k+1} \end{pmatrix} \right) / \left( \begin{pmatrix} 0 \\ s_k \end{pmatrix}, \mathcal{P}^{-1} \begin{pmatrix} 0 \\ s_k \end{pmatrix} \right)$$

$$\beta_k = \frac{(r_{k+1}, z_{k+1})}{(r_k, z_k)}$$

$$\begin{pmatrix} p_{k+1}^{(x)} \\ p_{k+1}^{(y)} \end{pmatrix} = \mathcal{P}^{-1} \begin{pmatrix} 0 \\ s_{k+1} \end{pmatrix} + \beta_k \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix} = \begin{pmatrix} -A^{-1} B p_{k+1}^{(y)} \\ p_{k+1}^{(y)} \end{pmatrix}$$

$$p_{k+1} = z_{k+1} + \beta_k p_k$$

Generic update:  $x_{k+1} = x_k + \alpha_k p_k^{(x)}$  with  $p_k^{(x)} = -A^{-1} B p_k^{(y)}$



## Saddle point problem and indefinite constraint preconditioner

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} I & B \\ B^T & 0 \end{pmatrix}$$

$$\mathcal{A}\mathcal{P}^{-1} = \begin{pmatrix} A(I - \Pi) + \Pi & (A - I)B(B^T B)^{-1} \\ 0 & I \end{pmatrix}$$

$\Pi = B(B^T B)^{-1} B^T$  - orth. projector onto  $\text{span}(B)$

[Lukšan, Vlček, 1998], [Gould, Keller, Wathen 2000]  
[Perugia, Simoncini, Arioli, 1999], [R, Simoncini, 2002]

## Indefinite constraint preconditioner: spectral properties

$\mathcal{AP}^{-1}$  **nonsymmetric** and **non-diagonalizable!**  
but it has a 'nice' spectrum:

$$\begin{aligned}\sigma(\mathcal{AP}^{-1}) &\subset \{1\} \cup \sigma(A(I - \Pi) + \Pi) \\ &\subset \{1\} \cup \sigma((I - \Pi)A(I - \Pi)) - \{0\}\end{aligned}$$

and only 2 by 2 Jordan blocks!

[Lukšan, Vlček 1998], [Gould, Wathen, Keller, 1999], [Perugia, Simoncini 1999]

## Krylov method with the constraint preconditioner: basic properties

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, r_0 = \begin{pmatrix} s_0 \\ 0 \end{pmatrix}, e_{k+1} = \begin{pmatrix} x - x_{k+1} \\ y - y_{k+1} \end{pmatrix}$$

$$r_{k+1} = \begin{pmatrix} f \\ 0 \end{pmatrix} - \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix}$$

$$\begin{aligned} r_0 = \begin{pmatrix} s_0 \\ 0 \end{pmatrix} &\Rightarrow r_{k+1} = \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix} \\ &\Rightarrow B^T(x - x_{k+1}) = 0 \\ &\Rightarrow x_{k+1} \in \text{Null}(B^T)! \end{aligned}$$

## Preconditioned CG method: error norm

$$r_{k+1}^T \mathcal{P}^{-1} r_j = 0, \quad j = 0, \dots, k$$

$x_{k+1}$  is an iterate from CG applied to

$$(I - \Pi)A(I - \Pi)x = (I - \Pi)f!$$

satisfying

$$\|x - x_{k+1}\|_A = \min_{u \in x_0 + \text{span}\{(I - \Pi)s_j\}} \|x - u\|_A$$

[Lukšan, Vlček 1998], [Gould, Wathen, Keller, 1999]



## Preconditioned CG algorithm

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, r_0 = b - \mathcal{A} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} s_0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} p_0^{(x)} \\ p_0^{(y)} \end{pmatrix} = \mathcal{P}^{-1} r_0 = \mathcal{P}^{-1} \begin{pmatrix} s_0 \\ 0 \end{pmatrix}$$

$k = 0, 1, \dots$

$$\alpha_k = \left( \begin{pmatrix} s_k \\ 0 \end{pmatrix}, \mathcal{P}^{-1} \begin{pmatrix} s_k \\ 0 \end{pmatrix} \right) / \left( \mathcal{A} \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix}, \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix} \right) \quad \alpha_k = (r_k, z_k) / (\mathcal{A} p_k, p_k)$$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \alpha_k \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix}$$

$$r_{k+1} = r_k - \alpha_k \mathcal{A} \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix} = \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix} \quad z_{k+1} = \mathcal{P}^{-1} r_{k+1}$$

$$\beta_k = \left( \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix}, \mathcal{P}^{-1} \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix} \right) / \left( \begin{pmatrix} s_k \\ 0 \end{pmatrix}, \mathcal{P}^{-1} \begin{pmatrix} s_k \\ 0 \end{pmatrix} \right) \quad \beta_k = (r_{k+1}, z_{k+1}) / (r_k, z_k)$$

$$\begin{pmatrix} p_{k+1}^{(x)} \\ p_{k+1}^{(y)} \end{pmatrix} = \mathcal{P}^{-1} \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix} + \beta_k \begin{pmatrix} p_k^{(x)} \\ p_k^{(y)} \end{pmatrix} \quad p_{k+1} = z_{k+1} + \beta_k p_k$$

## Preconditioned CG method: residual norm

$$\|x_{k+1} - x\| \rightarrow 0$$

but in general

$$y_{k+1} \not\rightarrow y$$

which is reflected in

$$\|r_{k+1}\| = \left\| \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix} \right\| \not\rightarrow 0!$$

but under appropriate scaling yes!

## Preconditioned CG method: residual norm

$$x_{k+1} \rightarrow x$$

$$x - x_{k+1} = \phi_{k+1}((I - \Pi)A(I - \Pi))(x - x_0)$$

$$s_{k+1} = \phi_{k+1}(A(I - \Pi) + \Pi)s_0$$

$$\sigma((I - \Pi)A(I - \Pi)) \sim \sigma(A(I - \Pi) + \Pi)?$$

$$\begin{aligned} \{1\} &\in \sigma((I - \Pi)A(I - \Pi)) - \{0\} \\ \Rightarrow \|r_{k+1}\| &= \left\| \begin{pmatrix} s_{k+1} \\ 0 \end{pmatrix} \right\| \rightarrow 0! \end{aligned}$$

## How to avoid misconvergence?

- ▶ Scaling by a constant  $\alpha > 0$  such that

$$\{1\} \in \text{conv}(\sigma((I - \Pi)\alpha A(I - \Pi)) - \{0\})$$

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \iff \begin{pmatrix} \alpha A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ \alpha y \end{pmatrix} = \begin{pmatrix} \alpha f \\ 0 \end{pmatrix}$$

$$v : \|(I - \Pi)v\| \neq 0, \quad \alpha = \frac{1}{((I - \Pi)v, A(I - \Pi)v)!}$$

- ▶ Scaling by a diagonal  $A \rightarrow (\text{diag}(A))^{-1/2} A (\text{diag}(A))^{-1/2}$  often gives what we want!
- ▶ Different direction vector  $p_k^{(y)}$  so that  $\|r_{k+1}\| = \|s_{k+1}\|$  is locally minimized!

$$y_{k+1} = y_k + (B^T B)^{-1} B^T s_k$$

[Braess, Deuffhard, Lipikov 1999], [Hribar, Gould, Nocedal, 1999], [Jiránek, R, 2008]

## Numerical experiments: a small model example

$$A = \text{tridiag}(1, 4, 1) \in \mathbb{R}^{25,25}, \quad B = \text{rand}(25, 5) \in \mathbb{R}^{25,5}$$
$$f = \text{rand}(25, 1) \in \mathbb{R}^{25}$$

$$\sigma(A) \subset [2.0146, 5.9854]$$

$$\alpha = 1/\tau \quad \sigma\left(\begin{pmatrix} \alpha A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} I & B \\ B^T & 0 \end{pmatrix}^{-1}\right)$$

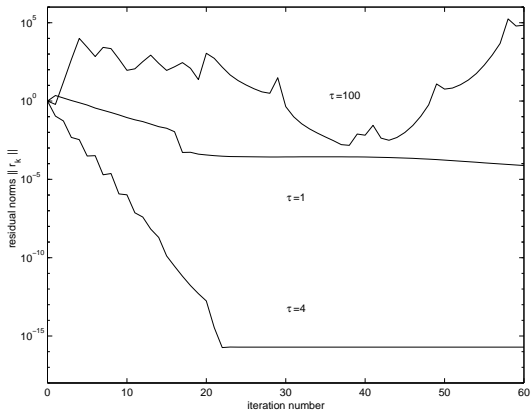
$$1/100 \quad [0.0207, 0.0586] \cup \{1\}$$

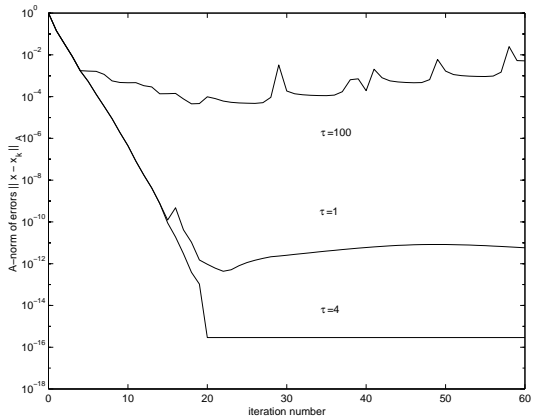
$$1/10 \quad [0.2067, 0.5856] \cup \{1\}$$

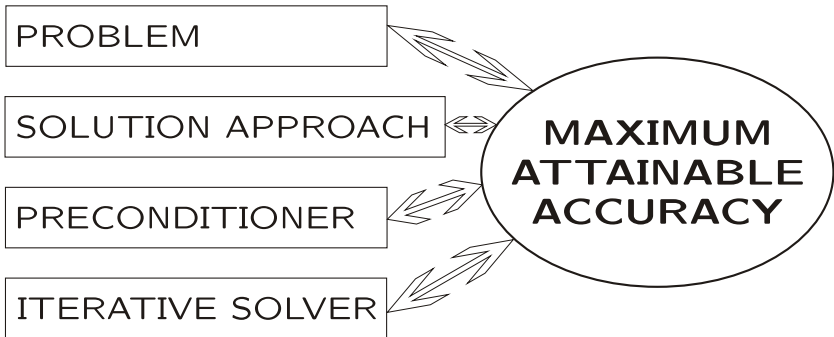
$$\mathbf{1/4} \quad [\mathbf{0.5170}, \mathbf{1.4641}]$$

$$1 \quad \{1\} \cup [2.0678, 5.8563]$$

$$4 \quad \{1\} \cup [8.2712, 23.4252]$$



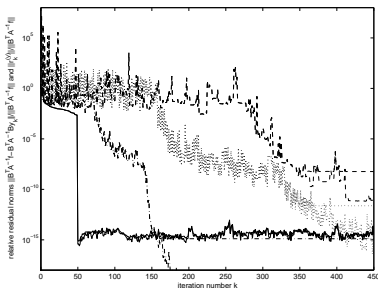






## Conclusions: segregated solution approach

- ▶ The accuracy measured by the residuals of the saddle point problem depends on the choice of the back-substitution scheme [Jiránek, R, 2008]. The schemes with (generic or corrected substitution) updates deliver approximate solutions which satisfy either the first or second block equation to working accuracy.
- ▶ Care must be taken when solving nonsymmetric systems [Jiránek, R, 2008], all bounds of the limiting accuracy depend on the maximum norm of computed iterates, cf. [Greenbaum 1994,1997], [Sleijpen, et al. 1994].



## Conclusions: coupled approach with indefinite preconditioner

- ▶ Short-term recurrence methods are applicable for saddle point problems with indefinite preconditioning at a cost comparable to that of symmetric solvers. There is a tight connection between the simplified Bi-CG algorithm and the classical CG.
- ▶ The convergence of CG applied to saddle point problem with indefinite preconditioner for all right-hand side vectors is not guaranteed. For a particular set of right-hand sides the convergence can be achieved by the appropriate scaling of the saddle point problem.
- ▶ Since the maximum attainable accuracy depends heavily on the size of computed residuals, a good scaling of the problems leads to approximate solutions satisfying both two block equations to the working accuracy.

## Thank you for your attention.

<http://www.cs.cas.cz/~miro>

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