Combination Preconditioning of saddle-point systems for positive definiteness

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joint work with Jen Pestana

Krylov subspace methods

for $\quad \mathcal{A}x = b$

compute iterates

$$x_k \in x_0 + \mathcal{K}_k(\mathcal{A}, r_0)$$

with residuals

$$b - \mathcal{A}x_k = r_k \in r_0 + \mathcal{A}\mathcal{K}_k(\mathcal{A}, r_0)$$

$$\text{ie.} \quad r_k = p(\mathcal{A})r_0, \quad p \in \Pi_k, p(0) = 1$$

since

$$\mathcal{K}_k(\mathcal{A}, r_0) = \text{span}\{r_0, \mathcal{A}r_0, \ldots, \mathcal{A}^{k-1}r_0\}$$

with some optimality or orthogonality condition.

Given an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

- Conjugate Gradients (*Hestenes & Stiefel (1952)*) computes iterates which minimize

$$\langle \boldsymbol{\mathcal{A}}(\boldsymbol{x} - \boldsymbol{x_k}), \boldsymbol{x} - \boldsymbol{x_k} \rangle$$

  when $\boldsymbol{\mathcal{A}}$ is self-adjoint and positive definite in $\langle \cdot, \cdot \rangle$

- MINRES (*Paige & Saunders (1975)*) computes iterates for which

$$\langle \boldsymbol{r_k}, \boldsymbol{r_k} \rangle$$

  is minimal when $\boldsymbol{\mathcal{A}}$ is self-adjoint in $\langle \cdot, \cdot \rangle$

- GMRES (*Saad & Schultz (1986)*) computes iterates for which

$$\langle \boldsymbol{r_k}, \boldsymbol{r_k} \rangle$$

  is minimal for general $\boldsymbol{\mathcal{A}}$

Conjugate Gradient Method (CG)

Choose x_0, compute $r_0 = b - \mathcal{A}x_0$, set $p_0 = r_0$

`for` $k = 0$ `until` convergence `do`

$\alpha_k = \langle r_k, r_k \rangle / \langle \mathcal{A}p_k, p_k \rangle$

$x_{k+1} = x_k + \alpha_k p_k$

$r_{k+1} = r_k - \alpha_k \mathcal{A}p_k$

<Test for convergence>

$\beta_k = \langle r_{k+1}, r_{k+1} \rangle / \langle r_k, r_k \rangle$

$p_{k+1} = r_{k+1} + \beta_k p_k$

`enddo`

computes iterates $\{x_k\}$ such that

$$\langle \mathcal{A}(x - x_k), x - x_k \rangle$$

is minimal when $\langle \mathcal{A}u, v \rangle = \langle u, \mathcal{A}v \rangle$ and $\langle \mathcal{A}u, u \rangle > 0$

Thus given any symmetric and positive definite matrix $\mathcal{H}$ if

$$\langle u, v \rangle = \langle u, v \rangle_{\mathcal{H}} = u^T \mathcal{H} v$$

Choose x_0, compute $r_0 = b - \mathcal{A}x_0$, set $p_0 = r_0$
`for` $k = 0$ `until` convergence `do`
$\alpha_k = \langle r_k, r_k \rangle_{\mathcal{H}} / \langle \mathcal{A}p_k, p_k \rangle_{\mathcal{H}}$
$x_{k+1} = x_k + \alpha_k p_k$
$r_{k+1} = r_k - \alpha_k \mathcal{A} p_k$
<Test for convergence>
$\beta_k = \langle r_{k+1}, r_{k+1} \rangle_{\mathcal{H}} / \langle r_k, r_k \rangle_{\mathcal{H}}$
$p_{k+1} = r_{k+1} + \beta_k p_k$
`enddo`

computes iterates $\{x_k\}$ such that

$$\langle \mathcal{A}(x - x_k), x - x_k \rangle_{\mathcal{H}} = (x - x_k)^T \mathcal{A}^T \mathcal{H} (x - x_k)$$

is minimal when $\langle \mathcal{A}u, v \rangle_{\mathcal{H}} = \langle u, \mathcal{A}v \rangle_{\mathcal{H}}$ and $\langle \mathcal{A}u, u \rangle_{\mathcal{H}} > 0$

Similarly for the MINRES method:

$v_0 = 0, w_0 = 0, w_1 = 0,$ choose x_0

Compute $r_0 = b - \mathcal{A}x_0,$ set $v_1 = r_0, \gamma_1 = \sqrt{\langle v_1, v_1 \rangle_{\mathcal{H}}}$

Set $\eta = \gamma_1, s_0 = s_1 = 0, c_0 = c_1 = 1$

`for` $j = 1$ `until` convergence `do`

$$v_j = v_j / \gamma_j$$
$$\delta_j = \langle v_j, \mathcal{A} v_j \rangle_{\mathcal{H}}$$
$$v_{j+1} = \mathcal{A} v_j - \delta_j v_j - \gamma_j v_{j-1}$$
$$\gamma_{j+1} = \sqrt{\langle v_{j+1}, v_{j+1} \rangle_{\mathcal{H}}}$$
$$\alpha_0 = c_j \delta_j - c_{j-1} s_j \gamma_j, \alpha_1 = \sqrt{\alpha_0^2 + \gamma_{j+1}^2}$$
$$\alpha_2 = s_j \delta_j + c_{j-1} c_j \gamma_j, \alpha_3 = s_{j-1} \gamma_j$$
$$c_{j+1} = \alpha_0 / \alpha_1; \; s_{j+1} = \gamma_{j+1} / \alpha_1$$
$$w_{j+1} = (v_{j+1} - \alpha_3 w_{j-1} - \alpha_2 w_j) / \alpha_1$$
$$x_j = x_{j-1} + c_{j+1} \eta w_{j+1}$$

$\eta = -s_{j+1} \eta,$ then <Test for convergence>

`enddo`

minimises $\langle r_k, r_k \rangle_{\mathcal{H}}$ when $\langle \mathcal{A}x, y \rangle_{\mathcal{H}} = \langle x, \mathcal{A}y \rangle_{\mathcal{H}}$

Self-adjointness: assume

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

is a symmetric bilinear form or an inner product

$\mathcal{A} \in \mathbb{R}^{n \times n}$ is self-adjoint in $\langle \cdot, \cdot \rangle$ iff

$$\langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}y \rangle \qquad \text{for all } x, y$$

Self-adjointness of $\mathcal{A}$ in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ $\quad (\langle x, y \rangle_{\mathcal{H}} = x^T \mathcal{H} y)$
thus means

$$x^T \mathcal{A}^T \mathcal{H} y = \langle \mathcal{A}x, y \rangle_{\mathcal{H}} = \langle x, \mathcal{A}y \rangle_{\mathcal{H}} = x^T \mathcal{H} \mathcal{A} y$$

for all $x, y \quad \Rightarrow$

$$\mathcal{A}^T \mathcal{H} = \mathcal{H} \mathcal{A}$$

is the relation for self-adjointness of $\mathcal{A}$ in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$

Preconditioning

For example left preconditioning:

$$\widehat{\mathcal{A}}x = \mathcal{P}^{-1}\mathcal{A}x = \mathcal{P}^{-1}b = \widehat{b}$$

induces a self-adjoint matrix in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ iff

$$\mathcal{A}^T\mathcal{P}^{-T}\mathcal{H} = \mathcal{H}\mathcal{P}^{-1}\mathcal{A}$$

An important example – The Bramble-Pasciak CG for saddle point problems:

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & -\boldsymbol{C} \end{bmatrix} \quad \text{with preconditioner} \quad \boldsymbol{\mathcal{P}} = \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{0} \\ \boldsymbol{B} & -\boldsymbol{I} \end{bmatrix}$$

The (left) preconditioned matrix

$$\widehat{\boldsymbol{\mathcal{A}}} = \boldsymbol{\mathcal{P}}^{-1}\boldsymbol{\mathcal{A}} = \begin{bmatrix} \boldsymbol{A}_0^{-1}\boldsymbol{A} & \boldsymbol{A}_0^{-1}\boldsymbol{B}^T \\ \boldsymbol{B}\boldsymbol{A}_0^{-1}\boldsymbol{A} - \boldsymbol{B} & \boldsymbol{B}\boldsymbol{A}_0^{-1}\boldsymbol{B}^T + \boldsymbol{C} \end{bmatrix}$$

is not symmetric but is self-adjoint and positive definite when

$$\boldsymbol{\mathcal{H}} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{A}_0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$$

defines an inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\boldsymbol{\mathcal{H}}} := \boldsymbol{x}^T \boldsymbol{\mathcal{H}} \boldsymbol{y}$

$\Rightarrow$ CG can be used in this inner product

Basic properties:

LEMMA If $\mathcal{A}_1$ and $\mathcal{A}_2$ are self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ then for any $\alpha, \beta \in \mathbb{R}$, $\alpha\mathcal{A}_1 + \beta\mathcal{A}_2$ is self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}}$

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and of relevance when preconditioning:

LEMMA For symmetric $\mathcal{A}$, $\widehat{\mathcal{A}} = \mathcal{P}^{-1}\mathcal{A}$ is self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ if and only if $\mathcal{P}^{-T}\mathcal{H}$ is self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{A}}$

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PROOF

$$(\mathcal{P}^{-T}\mathcal{H})^T\mathcal{A}=\mathcal{H}\mathcal{P}^{-1}\mathcal{A}=(\mathcal{P}^{-1}\mathcal{A})^T\mathcal{H}=\mathcal{A}(\mathcal{P}^{-T}\mathcal{H})$$

also combining the above:

LEMMA If $\mathcal{P}_1$ and $\mathcal{P}_2$ are left preconditioners for the symmetric matrix $\mathcal{A}$ for which symmetric matrices $\mathcal{H}_1$ and $\mathcal{H}_2$ exist with $\mathcal{P}_1^{-1}\mathcal{A}$ self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$ and $\mathcal{P}_2^{-1}\mathcal{A}$ self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}_2}$ and if for any α,β

$$\alpha\mathcal{P}_1^{-T}\mathcal{H}_1 + \beta\mathcal{P}_2^{-T}\mathcal{H}_2 = \mathcal{P}_3^{-T}\mathcal{H}_3$$

for some matrix $\mathcal{P}_3$ and some symmetric matrix $\mathcal{H}_3$ then $\mathcal{P}_3^{-1}\mathcal{A}$ is self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}_3}$.

shows: if we have two instances of this structure and can find such a splitting we have found a new preconditioner and a bilinear form in which the matrix is self-adjoint.

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for some matrix $\mathcal{P}_3$ and some symmetric matrix $\mathcal{H}_3$ then $\mathcal{P}_3^{-1}\mathcal{A}$ is self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{H}_3}$.

shows: if we have two instances of this structure and can find such a splitting we have found a new preconditioner and a bilinear form in which the matrix is self-adjoint.

$\Rightarrow$ Combination Preconditioners

Intriguing possibilities:

- if $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are all positive definite (so define inner products)
  but $\mathcal{P}_1^{-1}\mathcal{A}, \mathcal{P}_2^{-1}\mathcal{A}$ are indefinite ,
  can $\mathcal{P}_3^{-1}\mathcal{A}$ be positive definite?

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- can $\mathcal{H}_1$ and $\mathcal{H}_2$ be indefinite but $\mathcal{H}_3$ be positive definite?

Answers: YES and YES

Saddle Point examples

Bramble-Pasciak CG (*Bramble & Pasciak (1988)*) widely used CG technique with preconditioner

$$\mathcal{P}^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ BA_0^{-1} & -I \end{bmatrix}$$

and inner product matrix

$$\mathcal{H} = \begin{bmatrix} A - A_0 & 0 \\ 0 & I \end{bmatrix}$$

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$$\mathcal{H} = \begin{bmatrix} A - A_0 & 0 \\ 0 & I \end{bmatrix}$$

main drawback: requires

$$A_0 < A$$

but $\mathcal{P}^{-1}\mathcal{A}$ is always positive definite when this is true

Examples: BP with Schur complement preconditioner
(*Klawonn (1998), Meyer et al. (2001), Simoncini (2001)*)

$$\mathcal{P}^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ S_0^{-1} B A_0^{-1} & -S_0^{-1} \end{bmatrix}$$

Inner product:

$$\mathcal{H} = \begin{bmatrix} A - A_0 & 0 \\ 0 & S_0 \end{bmatrix}$$

similar conditions as BP for positive definiteness

Examples: Zulehner (*Zulehner (2001), Schöberl & Zulehner (2007)*)

$$\mathcal{P} = \begin{bmatrix} A_0 & B^T \\ B & BA_0^{-1}B^T - S_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ BA_0^{-1} & I \end{bmatrix} \begin{bmatrix} A_0 & B^T \\ 0 & -S_0 \end{bmatrix}$$

gives $\mathcal{P}^{-1}\mathcal{A}$ self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$,

$$\mathcal{H} = \begin{bmatrix} A_0 - A & 0 \\ 0 & BA_0^{-1}B^T - S_0 \end{bmatrix}$$

So $\mathcal{H}$ defines an inner product if $A_0 > A$ and $S_0 < BA_0^{-1}B^T$

Whenever $\mathcal{H}$ is positive definite, then $\mathcal{P}^{-1}\mathcal{A}$ is positive definite in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

Examples: Benzi-Simoncini *(Benzi and Simoncini (2006))* extension of CG method of *Fischer, Ramage, Silvester & W (1998)*

$$\mathcal{P}^{-1} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix}$$

inner product:

$$\mathcal{H} = \begin{bmatrix} \boldsymbol{A} - \gamma \boldsymbol{I} & \boldsymbol{B}^T \\ \boldsymbol{B} & \gamma \boldsymbol{I} \end{bmatrix}$$

Extension for $\boldsymbol{C} \neq \boldsymbol{0}$ *(Liesen (2006), Liesen & Parlett (2007))*:

$$\mathcal{H} = \begin{bmatrix} \boldsymbol{A} - \gamma \boldsymbol{I} & \boldsymbol{B}^T \\ \boldsymbol{B} & \gamma \boldsymbol{I} - \boldsymbol{C} \end{bmatrix}$$

Example: Bramble-Pasciak$^+$ method (BP$^+$)*(Stoll & W(2008))*

$$\mathcal{P}^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ BA_0^{-1} & I \end{bmatrix}$$

and inner product

$$\mathcal{H} = \begin{bmatrix} A + A_0 & 0 \\ 0 & I \end{bmatrix}$$

Note: $\mathcal{H}$ defines an inner product for any symmetric and positive definite preconditioner A_0
$\Rightarrow$ can always apply MINRES in this inner product

Example: Bramble-Pasciak^{+} method (BP^{+})*(Stoll & W(2008))*

$$\mathcal{P}^{-1} = \begin{bmatrix} \boldsymbol{A_0^{-1}} & \mathbf{0} \\ \boldsymbol{BA_0^{-1}} & \boldsymbol{I} \end{bmatrix}$$

and inner product

$$\mathcal{H} = \begin{bmatrix} \boldsymbol{A} + \boldsymbol{A_0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix}$$

Note: $\mathcal{H}$ defines an inner product for any symmetric and positive definite preconditioner $\boldsymbol{A_0}$
$\Rightarrow$ can always apply MINRES in this inner product

But preconditioned matrix always indefinite in this inner product

Similarly there exists a Schöberl-Zulehner$^+$ method (SZ$^+$) *(Pestana & W(2012))*

$$\mathcal{P} = \begin{bmatrix} A_0 & -B^T \\ -B & BA_0^{-1}B^T + S_0 \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ -BA_0^{-1} & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & -S_0 \end{bmatrix} \begin{bmatrix} I & -A_0^{-1}B^T \\ 0 & I \end{bmatrix}$$

gives $\mathcal{P}^{-1}\mathcal{A}$ self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}}$,

$$\mathcal{H} = \begin{bmatrix} A_0 + A & 0 \\ 0 & BA_0^{-1}B^T + S_0 \end{bmatrix}$$

So $\mathcal{H}$ always defines an inner product, but $\mathcal{P}^{-1}\mathcal{A}$ is always indefinite in this inner product.

final example: Block Diagonal Preconditioner (BD) *(Silvester& W (1993), Murphy, Golub & W (2000), Korzak (1999), Kuznetsov (1995))*

$$\mathcal{P} = \mathcal{H} = \begin{bmatrix} A_0 & 0 \\ 0 & S_0 \end{bmatrix}$$

for which

$$\mathcal{H}\mathcal{P}^{-1}\mathcal{A} = \mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}$$

is clearly symmetric.

But $\mathcal{A}$ indefinite $\Rightarrow \mathcal{P}^{-1}\mathcal{A}$ is always indefinite in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$

Many of the above are special cases of the Krzyzanowski preconditioner *(Krzyzanowski (2011))*

$$\mathcal{P} = \begin{bmatrix} I & 0 \\ cBA_0^{-1} & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & -S_0 \end{bmatrix} \begin{bmatrix} I & dA_0^{-1}B^T \\ 0 & I \end{bmatrix}$$

for which $\mathcal{P}^{-1}\mathcal{A}$ is self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ with

$$\mathcal{H} = \epsilon \begin{bmatrix} A_0 - cA & 0 \\ 0 & S_0 + cdBA_0^{-1}B^T + dC \end{bmatrix}, \epsilon = \pm 1$$

General (but not simple in general) formulae for eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ are available *(Pestana & W (2012))*

Combination preconditioning

Final lemma above shows that if can find $\mathcal{P}_3$ and $\mathcal{H}_3$ with

$$\alpha \mathcal{P}_1^{-T} \mathcal{H}_1 + \beta \mathcal{P}_2^{-T} \mathcal{H}_2 = \mathcal{P}_3^{-T} \mathcal{H}_3$$

this gives a new preconditioner $\mathcal{P}_3$ and the symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{H}_3}$ in which $\mathcal{P}_3^{-1} \mathcal{A}$ is self-adjoint

Combine Bramble-Pasciak and Benzi-Simoncini:

$$\alpha \mathcal{P}_1^{-1}\mathcal{H}_1 + \beta \mathcal{P}_2^{-1}\mathcal{H}_2 =$$

$$\begin{bmatrix} (\alpha A_0^{-1} + \beta I)A - (\alpha + \beta\gamma)I & (\alpha A_0^{-1} + \beta I)B^T \\ -\beta B & -(\alpha + \beta\gamma)I \end{bmatrix}$$

One possibility for $\alpha \mathcal{P}_1^{-T}\mathcal{H}_1 + \beta \mathcal{P}_2^{-T}\mathcal{H}_2 = \mathcal{P}_3^{-T}\mathcal{H}_3$ is

$$\mathcal{P}_3^{-T} = \begin{bmatrix} \alpha A_0^{-1} + \beta I & 0 \\ 0 & -\beta I \end{bmatrix}$$

and

$$\mathcal{H}_3 = \begin{bmatrix} A - (\alpha + \beta\gamma)(\alpha A_0^{-1} + \beta I)^{-1} & B^T \\ B & \frac{\alpha+\beta\gamma}{\beta} I \end{bmatrix}$$

Combine BP$^-$ and BP$^+$

$$\alpha \mathcal{P}_1^{-T} \mathcal{H}_1 + (1-\alpha) \mathcal{P}_2^{-T} \mathcal{H}_2 =$$

$$\begin{bmatrix} A_0^{-1} A + (1-2\alpha) I & A_0^{-1} B^T \\ 0 & (1-2\alpha) I \end{bmatrix}$$

Combine BP$^-$ and BP$^+$

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can be split as

$$\mathcal{P}_3^{-T} = \begin{bmatrix} A_0^{-1} & A_0^{-1}B^T \\ 0 & (1-2\alpha)I \end{bmatrix}, \mathcal{H}_3 = \begin{bmatrix} A + (1-2\alpha)A_0 & 0 \\ 0 & I \end{bmatrix}$$

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Recall $\mathcal{P}_3^{-1} \mathcal{A}$ is self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{H}_3}$

Combine BP$^-$ and BP$^+$

$$\alpha \mathcal{P}_1^{-T} \mathcal{H}_1 + (1-\alpha) \mathcal{P}_2^{-T} \mathcal{H}_2 =$$

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Recall $\mathcal{P}_3^{-1} \mathcal{A}$ is self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{H}_3}$

($\alpha = 1 \leftrightarrow$ BP$^-$, $\quad \alpha = 0 \leftrightarrow$ BP$^+$)

Combine BP$^+$ and SZ$^+$

$$\mathcal{P}_3 = \begin{bmatrix} I & 0 \\ -BA_0^{-1} & I \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha+\beta} A_0 & \frac{-\beta}{\alpha+\beta} B^T \\ 0 & S_0 \end{bmatrix}$$

$$\mathcal{H}_3 = \begin{bmatrix} A + A_0 & 0 \\ 0 & (\alpha+\beta) S_0 + \beta B A_0^{-1} B^T \end{bmatrix}$$

Clearly $\mathcal{H}_3$ positive definite at least for some α, β

but $\mathcal{P}_3^{-1}\mathcal{A}$ always indefinite when $\langle \cdot, \cdot \rangle_{\mathcal{H}_3}$ defines an inner product.

Combine BP$^+$ and BD

$$\mathcal{P}_3 = \begin{bmatrix} A_0 & 0 \\ -\frac{\alpha}{\alpha+\beta}B & \frac{1}{\alpha+\beta}S_0 \end{bmatrix},$$

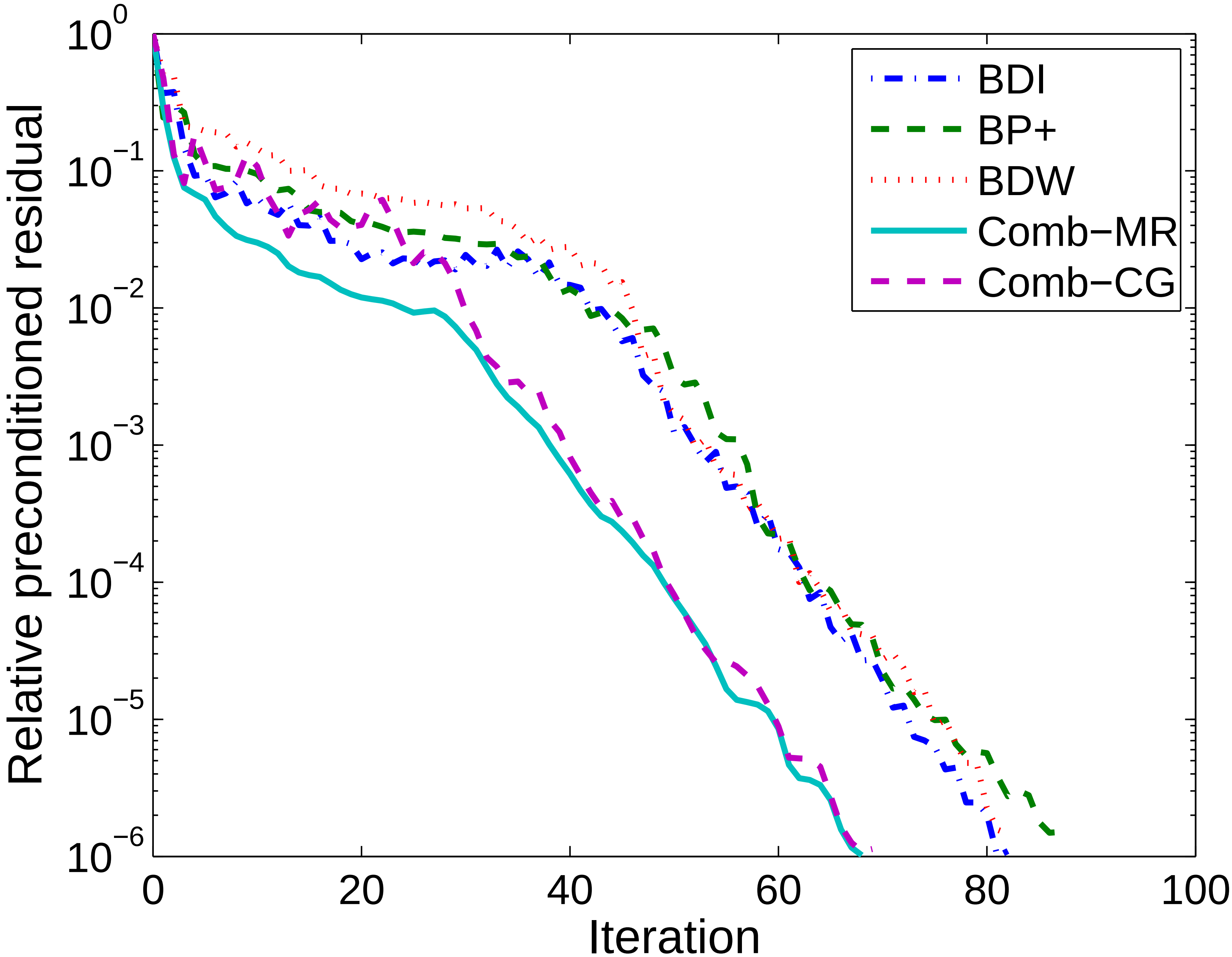
$$\mathcal{H}_3 = \begin{bmatrix} \alpha(A + A_0) + \beta A_0 & 0 \\ 0 & S_0 \end{bmatrix}$$

Theorem: if $\alpha > 0$ and $\alpha + \beta < 0$ then $\mathcal{H}_3$ is positive definite and so $\langle \cdot, \cdot \rangle_{\mathcal{H}_3}$ defines an inner product with respect to which
$\mathcal{P}_3^{-1}\mathcal{A}$ is positive definite if and only if

$$A_0 > \frac{-\alpha}{\alpha + \beta} A$$

CG iteration counts for the 4 standard ifiss Stokes test problems: Taylor-Hood elements ($C = 0$), A_0: no-fill `ichol`, S_0: mass matrix

Problem	h	BP$^+$	BD	Comb (α, β)	% reduction
Channel flow	2^{-3}	41	38	27 (1.3,-2)	29
	2^{-4}	59	57	43 (1.7,-2)	25
	2^{-5}	95	95	86 (0.7,-0.6)	9
Backward step	2^{-3}	57	55	41 (1.4,-2)	25
	2^{-4}	88	83	69 (1.4,-1.6)	17
	2^{-5}	147	148	140 (1.2,-1)	5
Regularized cavity	2^{-3}	34	32	21(1.1,-1.8)	34
	2^{-4}	52	48	40 (1.2,-1.5)	17
	2^{-5}	88	81	73 (1.9,-2)	10
Colliding flow	2^{-3}	28	28	20(1.1,-1.8)	29
	2^{-4}	46	41	34 (0.8,-1)	17
	2^{-5}	72	71	56 (1.4,-1.5)	21



Relative preconditioned residual
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
20
40
60
80
100
Iteration
BDI
BP+
BDW
Comb−MR
Comb−CG

CG iteration counts for the 4 standard ifiss Stokes test problems: Taylor-Hood elements ($\boldsymbol{C = 0}$), $\boldsymbol{A_0}$: 1 AMG V-cycle, $\boldsymbol{S_0}$: mass matrix

Problem	$\boldsymbol{h}$	BP$^+$	BD	Comb ($\boldsymbol{\alpha, \beta}$)	% reduction
Channel flow	$\mathbf{2^{-3}}$	31	29	18 (1.1,-2)	38
	$\mathbf{2^{-4}}$	36	33	19 (1.1,-2)	42
	$\mathbf{2^{-5}}$	39	34	20 (1.1,-2)	41
Backward step	$\mathbf{2^{-3}}$	47	43	25 (1.1,-2)	42
	$\mathbf{2^{-4}}$	52	48	28 (1.1,-2)	42
	$\mathbf{2^{-5}}$	53	50	28 (1.1,-2)	44
Cavity flow	$\mathbf{2^{-3}}$	30	26	15 (1.1,-2)	42
	$\mathbf{2^{-4}}$	34	30	18 (1.1,-2)	40
	$\mathbf{2^{-5}}$	35	32	18 (1.1,-2)	44
Colliding flow	$\mathbf{2^{-3}}$	23	24	15 (1.1,-2)	35
	$\mathbf{2^{-4}}$	29	26	17 (1.1,-2)	35
	$\mathbf{2^{-5}}$	29	28	18 (1.1,-2)	36

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- two indefinite examples can be combined to give a positive definite preconditioned matrix (and so allow CG in the associated inner product)
- application here to saddle-point matrices, but theory is more general

Acknowledgement

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