A shifted preconditioner for the damped Helmholtz equation

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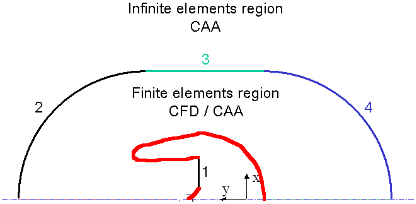
Outline

1. Shifted preconditioner for damped Helmholtz (NLAA 2009)

2. IMF: an Incomplete Multifrontal Factorization

Motivation: aero-acoustics



- Liner for reducing noise emitted to the far field
- Differential algebraic equation (in frequency domain)

$$(K + i\omega C - \omega^2 M)x = f$$

with

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad C = \begin{bmatrix} C_1 & C_{1,2} \\ C_{2,1} & C_2 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} K_1 & K_{1,2} \\ K_{2,1} & K_2 \end{bmatrix} .$$

Shifting

- When $C = 0$, shifting is effective, i.e. solve:

$$(K - \omega^2 M + \gamma M)^{-1}(K - \omega^2 M)x = (K - \omega^2 M + \gamma M)^{-1} f$$

  [Magolu monga Made et al. 2000] [Bayliss et al. 1983] [Laird, Giles, 2002] [Erlangga et al. 2004, 2006, 2006]

- When $C \neq 0$, 'linearize' to

$$(i\omega B + A)\mathbf{v} = \mathbf{f}$$

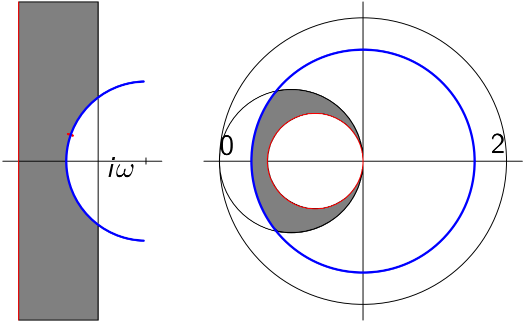
$$A = \begin{bmatrix} K + i\omega C - \omega^2 M & 0 \\ 0 & -I \end{bmatrix} \qquad B = \begin{bmatrix} C + i2\omega M & M \\ I & 0 \end{bmatrix}$$

- Solve shifted problem:

$$(B + \delta A)^{-1} A\mathbf{v} = (B + \delta A)^{-1}\mathbf{f}$$

Spectral properties

- Preconditioned matrix $(B + \delta A)^{-1} A$



λ plane $\qquad$ θ plane

- Spectral mapping of quadratic eigenvalue problem $\det(K + \lambda C + \lambda^2 M) = 0$ to eigenvalues of $(B + \delta A)^{-1} A$
- $\theta = \frac{\lambda - i\omega}{\lambda - (i\omega + \delta^{-1})}$
- Eigenvalues closer to 1 when δ is larger

Iterative process

- Preconditioned system:

$$(B+\delta A)^{-1}A\mathbf{v} \quad = \quad (B+\delta A)^{-1}\mathbf{f}$$

- Compare this with

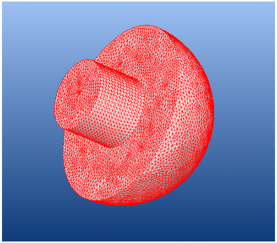
$$P^{-1}Zx = P^{-1}f$$

  with

$$\begin{aligned} Z &= K + i\omega C - \omega^2 M \\ P &= \delta^{-2}M + \delta^{-1}(C + i2\omega M) + Z \end{aligned}$$

- The field of values of $P^{-1}Z$ is contained by the field of values of $(B+\delta B)^{-1}A$

'Mushroom' problem



- Complex unsymmetric problem: infinite elements on spherical part of the boundary (8, 764 infinite elements of order 5)
- Comparison for different values of $\delta\omega$:
  [Erlangga, Vuik, Oosterlee, 2006], [Erlangga, Oosterlee, Vuik, 2006]
  - $\delta\omega = 1$ and $\delta\omega = 2$.
  - $\delta\omega = 1 + \sqrt{1 - i}$
- Results using BiCGStab with ILU applied to P

'Mushroom' problem

- $\omega = 500$, $kr = 9.13$.

$\delta\omega$	BiCGStab – ILU(1) iterations	BiCGStab – ILU(1) time (s)	BiCGStab – ILU(2) iterations	BiCGStab – ILU(2) time (s)
0.2	no cvg		no cvg	
0.5	1590	81	1713	164
1	727	38	608	55
2	465	25	298	29
5	904	46	184	19
7	2455	125	247	24
∞	no cvg		no cvg	

- Same problem, $\delta\omega = \alpha^{-1}(1 + \sqrt{1 - \alpha i})$

α	$\lvert\omega\delta\rvert$	BiCGStab – ILU(1) iterations	BiCGStab – ILU(1) time (s)	BiCGStab – ILU(2) iterations	BiCGStab – ILU(2) time (s)
2	1.2	497	26	484	43
1	2.1	378	20	267	25
0.5	4.0	612	31	184	19
0.2	10.			767	67

'Mushroom' problem

- Same problem, $\delta\omega = (1 + \sqrt{1 - i})$

ω	BiCGStab – ILU(1) iterations	BiCGStab – ILU(1) time (s)	BiCGStab – ILU(2) iterations	BiCGStab – ILU(2) time (s)
100	253	13	155	16
200	456	24	172	18
300	548	30	235	23
400	393	21	207	21
500	372	20	267	25
600	563	29	487	44
700	429	23	336	32

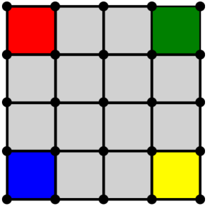
Incomplete multifrontal factorization

- Large sparse linear system $Ax = b$ where A is given in finite element format:

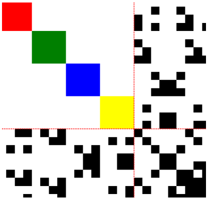
$$A = \sum_{e \in \mathcal{E}} P_e A_e P_e^T$$

  - A_e is dense (full) matrix
  - P_e selects degrees of freedom associated with the element

- Multilevel preconditioning using element structure instead of sparse.



$\mathcal{D}_k \equiv \langle$■, ■, ■, ■$\rangle$



$D_k \equiv \mathrm{diag}($■, ■, ■, ■$)$

Incomplete multilevel factorization

We propose IMF(κ), a multilevel block-ILU factorization:

$$M_k' := P_k M_k P_k^T = \begin{bmatrix} D_k & U_k \\ L_k & S_k \end{bmatrix} \approx \begin{bmatrix} I & 0 \\ L_k D_k^{-1} & I \end{bmatrix} \begin{bmatrix} D_k & U_k \\ 0 & M_{k+1} \end{bmatrix},$$

computed in a multifrontal manner, where

- k the current level subscript,
- P_k is a partitioning into fine/coarse,
- D_k is a block diagonal matrix, and
- the reduced system $M_{k+1} = \begin{cases} S_k - L_k D_k^{-1} U_k & \text{if } k < \kappa \\ S_k - L_k D_k^{-1} U_k + E_k & \text{if } k \geq \kappa \end{cases}$

Incomplete multilevel factorization

IMF is a block-ILU with multilevel structure:

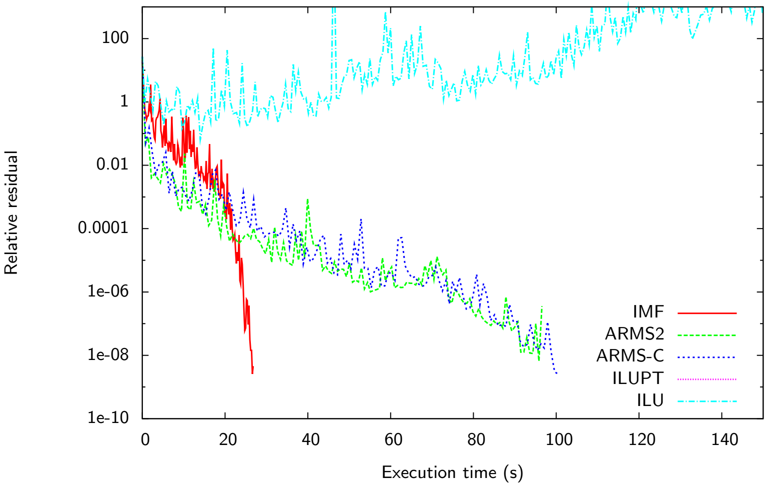
$$M_{\text{IMF}} = \left[\begin{array}{c|c} I & \\ \hline L_0 D_0^{-1} & \begin{array}{c|c} I & \\ \hline L_1 D_1^{-1} & \begin{array}{c|c} \ddots & \\ \hline \cdots & \begin{array}{c|c} I & \\ \hline L_f D_f^{-1} & I \end{array} \end{array} \end{array} \end{array}\right] \left[\begin{array}{c|c} D_0 & U_0 \\ \hline & \begin{array}{c|c} D_1 & U_1 \\ \hline & \begin{array}{c|c} \ddots & \vdots \\ \hline & \begin{array}{c|c} D_f & U_f \\ \hline & D_{f+1} \end{array} \end{array} \end{array} \end{array}\right]$$

Structural dynamics

Problem	Precond.	Parameter	ϕ	T_f	T_s	υ	ρ
SHIPSEC1	IMF	0	1.35	2.06	20.20	781	$7.36 \cdot 10^{-13}$
		1	2.61	3.29	13.04	408	$1.87 \cdot 10^{-12}$
	ARMS2	$7.12500 \cdot 10^{-3}$	1.34	4.50	$*$	$*$	$1.11 \cdot 10^{-06}$
		$2.51600 \cdot 10^{-3}$	2.64	8.30	$*$	$*$	$9.65 \cdot 10^{-04}$
	ARMS-C	$3.76399 \cdot 10^{-2}$	1.36	4.44	$*$	$*$	—
		$2.84950 \cdot 10^{-2}$	2.66	11.29	$*$	$*$	$8.00 \cdot 10^{+02}$
	ILUTP	$2.12891 \cdot 10^{-1}$	1.36	4.60	$*$	$*$	$1.05 \cdot 10^{+09}$
		$1.89304 \cdot 10^{-1}$	2.60	11.43	$*$	$*$	—
SHIPSEC5	IMF	0	1.33	2.71	19.64	574	$2.61 \cdot 10^{-12}$
		1	2.67	4.66	16.50	372	$8.59 \cdot 10^{-13}$
	ARMS2	$7.88000 \cdot 10^{-3}$	1.33	6.22	$*$	$*$	$6.26 \cdot 10^{-05}$
		$2.10000 \cdot 10^{-3}$	2.67	11.29	$*$	$*$	$2.24 \cdot 10^{-04}$
	ARMS-C	$3.82300 \cdot 10^{-2}$	1.32	6.00	$*$	$*$	—
		$2.60410 \cdot 10^{-2}$	2.67	15.66	$*$	$*$	—
	ILUTP	$2.12600 \cdot 10^{-1}$	1.32	6.21	$*$	$*$	$2.12 \cdot 10^{+16}$
		$1.88800 \cdot 10^{-1}$	2.68	16.90	$*$	$*$	—

Convergence

2D Navier-Stokes equation, lid driven cavity, Reynolds number 3500, 16 128 elems (Q1-Q1), 49 440 unknowns, 1 243 530 nnzs. Source: IFISS.



Computational performance

2D Navier-Stokes equation, lid driven cavity, Reynolds number 3500, 16 128 elems (Q1-Q1), 49 440 unknowns, 1 243 530 nnzs. Source: IFISS.