The parameterized Lanczos method for multiple right-hand sides

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Outline

(joint work with Zhaojun Bai)

Motivation

Fast alternatives
  Modal truncation
  AMLS
  Lanczos method

Further analysis

Applications

Numerical example

Conclusions
Vibration problems

A car window is subjected to vibrations from outside, including wind. Glass manufacturers want to compute the transmission of noise through windscreens.
Fourier analysis

- Solve

\[(K - \omega^2 M)x = f\]

with \(K, M: \text{n} \times \text{n}\) sparse matrices, symmetric.

- \(\omega \in \Omega = [\omega_{\text{min}}, \omega_{\text{max}}]\).
  We also define \(\Omega^2 = [\omega^2_{\text{min}}, \omega^2_{\text{max}}]\).

- \(x\) is called the frequency response function.

- Important is speed, not good reduction.
Alternatives: modal truncation

- Consider the eigendecomposition

\[ Ku_j = \lambda_j Mu_j \]

- The solution of \((K - \omega^2 M)x = f\) is

\[ x = \sum_{j=1}^{n} u_j \frac{u_j^* f}{\lambda_j - \omega^2} \]

- Rational function with poles \(\lambda_j\).
Alternative: modal truncation

\[ x = \sum_{j=1}^{n} u_j \frac{u_j^* f}{\lambda_j - \omega^2} \approx \sum_{j=1}^{k} u_j \frac{u_j^* f}{\lambda_j - \omega^2} \]
Alternative: AMLS

[Bennighof & Kaplan, 1998], [Ko & Bai, 2007]

The physical problem is split in e.g. two domains, each of which are approximated by the local modes. This leads to the following system in local modal coordinates

\[
\begin{bmatrix}
K_1 & K_2 \\
K_2 & K_3
\end{bmatrix} - \omega^2 \begin{bmatrix}
M_1 & M_{13} \\
M_{31} & M_{23}
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
\]

where \( K_s \) are diagonal matrix and \( x_s \) and \( f_s \) are modal coefficients for \( x \) and \( f \).
Alternative: AMLS

The solution of

$$(K - \omega^2 M)x = f$$

for $\omega \in \Omega$ is split into two parts.

- Let $U_p = [u_1, \ldots, u_p]$ be the eigenvectors corresponding to the eigenvalues in $\Omega^2$.
- Solve the $p \times p$ problem

$$U_p^*(K - \omega^2 M)U_p z = U_p^* f$$

- Solve the ‘rest’ system

$$(I - MU_p U_p^*)(K - \omega^2 M)(I - MU_p U_p^*)y = (I - MU_p U_p^*) f$$

- $x = U_p z + y$
Alternative: AMLS

- Preconditioner for the ‘rest’ system:
  \[
  (I - MU_p U_p^*)(K - \sigma M)^{-1}(I - MU_p U_p^*)
  \]

- Preconditioned system is
  \[
  By = b
  \]

with

\[
\begin{align*}
b &= (I - MU_p U_p^*)(K - \sigma M)^{-1}f \\
B &= (I - MU_p U_p^*)(K - \sigma M)^{-1}(K - \omega^2 M)(I - MU_p U_p^*)
\end{align*}
\]
Alternative: AMLS

- Spectrum of $Ku = \lambda Mu$:
  $$\lambda_1, \lambda_2, \ldots, \lambda_n$$

- Spectrum of $(I - B)$:
  $$\theta_j = 1 \text{ for } j = 1, \ldots, p$$
  $$\theta_j = \frac{\omega^2 - \sigma}{\lambda_j - \sigma} \text{ for } j > p.$$  
  $$|\theta_j| < 1 \text{ if } j > p, \lambda_1, \ldots, \lambda_p \text{ are the eigenvalues in } \Omega^2 \text{ and } \sigma \in \Omega^2$$

- Convergence of stationary method
Alternative: Lanczos process

- Use a Krylov method for solving \( By = b \)
- Since \( x^* MBy = y^* MBx \), we use the Lanczos method with \( M \) inner product

Lanczos method:
1. Let \( v_1 = b/\|b\|_M \)
2. For \( j = 1, \ldots, k \)
   2.1. Compute Krylov vector \( v_{j+1} = Bv_j \).
   2.2. Orthogonalize \( v_{j+1} \) against \( v_1, \ldots, v_j \)
   so that \( v_{j+1}^* Mv_j = 0 \).

- Lanczos vectors \( V_k = [v_1, \ldots, v_k] \).
- Recurrence relation: \( BV_k - V_k T_k = \beta_k v_{k+1} e_k^* \) with \( T_k \) tridiagonal
- \( y = V_k T_k^{-1} e_1 \|b\|_M \)
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Shifted Lanczos method

- Recall that

\[ B = (I - MU_p U_p^*)(K - \sigma M)^{-1}(K - \omega^2 M) \]

- The Krylov space of \( B \) lies in \( \{U_p\} \perp_M \) and is constructed by orthogonalization against \( U_p \):

\[ (K - \sigma M)^{-1}(K - \omega^2 M)V_k - V_k T_k = \beta_k v_{k+1} e_k^* \]

with \( U_p^* MV_k = 0 \)

- \( V_k \) and \( T_k \) can be computed from the shifted Lanczos method.

\[ (K - \sigma M)^{-1} M \tilde{V}_k - \tilde{V}_k \tilde{T}_k = \beta_k \tilde{v}_{k+1} e_k^* \]

- \( V_k = \tilde{V}_k \) and \( T_k = I + (\sigma - \omega^2) \tilde{T}_k \)
Shifted linear systems

- Analyzed in the context of model reduction methods
- Feldman, Freund, Bai, Grimme, Sorensen, Van Dooren, Ruhe, Skoogh, Olsson, Simoncini, M., ...
- Connection with eigendecomposition
- Connection with iterative linear solvers
- Connection with rational approximation (Padé)
Convergence

- If $\sigma \in \Omega^2$ and $\lambda_1, \ldots, \lambda_p \in \Omega^2$, then $B$ is a positive definite operator.

- The more eigenvalues outside $\Omega^2$ are included in $\lambda_1, \ldots, \lambda_p$, the more clustered the eigenvalues of $B$ become, so faster convergence.
Required accuracy

- Let \( \theta_j \) be the eigenvalues of \( B \)
- Let \((\hat{\theta}_j, \hat{u}_j)\) be Ritz pairs of

\[
A = (K - \sigma M)^{-1}(K - \omega^2 M)
\]

and

\[
\hat{B} = (I - M\hat{U}_p\hat{U}_p^*)(K - \sigma M)^{-1}(K - \omega^2 M)(I - M\hat{U}_p\hat{U}_p^*)
\]

with

\[
\|A\hat{u}_j - \hat{u}_j\hat{\theta}_j\|_M \leq \xi
\]

for \( j = 1, \ldots, p \), then

\[
\kappa(\hat{B}) \leq \kappa(B) \frac{\max |\theta_j| + \xi}{\max |\theta_j|} \frac{\min |\theta_j|}{\min |\theta_j - \xi|}
\]

where \( j = p + 1, \ldots, n \)

- Noting that most \( \theta_j \) are close to one, \( \xi \) need not be very small for \( \kappa(\hat{B}) \simeq \kappa(B) \).
Eigenvalue solver

- Ritz values: $T_k z = \theta z$
- Ritz vectors: $\hat{u} = V_k z$
Convergence of Ritz vectors

- Eigenvalues in $\Omega^2$ are computed fairly accurately

- If $\|B\hat{y} - b\|_M \leq \gamma \|y\|_M$ for all $\omega \in \Omega$
- then $\rho_j = \|B\hat{u}_j - \hat{u}_j\hat{\theta}_j\|_M$ with

$$\rho_j \leq \frac{\gamma}{|\lambda_j - \sigma|}$$
Applications

- AMLS frequency sweeping
- Multiple right-hand sides:
  - Parameterized Lanczos for right-hand side 1
  - Keep Ritz vectors
  - Recycle Ritz vectors for coming right-hand sides
- Changing $\sigma$: recycle Ritz vectors for new pole.
Glaverbel-BMW windscreen

- grid: 3 layers of $60 \times 30$ HEX08 elements ($n = 22,692$)
- $\Omega = [0, 100]$ 
- First run:
  - unit point force at one of the corners
  - Use Lanczos method with $k = 20$ vectors.
  - We keep the Ritz values in $[0, 2 \times 100^2]$ : $p = 14$
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Windscreen

- Second run with other right-hand side
  - Perform 6 additional Lanczos steps

![Graph showing multiple curves with annotations](attachment:image.png)

- The largest $\kappa(\hat{B})$ is 1.9813.
- Six iterations reduce the error in the $M\hat{B}$ norm by $2 \cdot 10^{-5}$. 
Conclusions

- Extension of recycling Ritz vectors of linear systems to parameterized linear systems.
- Similar analysis for unsymmetric matrices