Introduction

Trajectory Piecewise Linear (TPWL)

Proper Orthogonal Decomposition for DAEs

Model reduction for nonlinear DAEs

A. Verhoeven\textsuperscript{1,2} \hspace{1cm} P. Astrid\textsuperscript{1} \hspace{1cm} T. Voss\textsuperscript{2}

\texttt{averhoev@win.tue.nl}

\textsuperscript{1}Eindhoven University of Technology (CASA)

\textsuperscript{2}Philips Electronic Design & Tools / Analogue Simulation

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# Outline

1. **Introduction**
   - Model reduction for nonlinear DAEs
   - Survey of existing nonlinear MOR algorithms
   - Formulation as optimization problem

2. **Trajectory Piecewise Linear (TPWL)**
   - Introduction to TPWL
   - Error control of TPWL

3. **Proper Orthogonal Decomposition for DAEs**
   - Description of method for ODEs
   - Application to DAEs
   - Missing Point Estimation
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Introduction

Consider the following nonlinear DAE system

\[ \frac{d}{dt} q(t, x) + j(t, x) = Bu, \quad x(0) = x_0 \]
\[ y = Dx \]

We are interested in the input-output behaviour of this nonlinear system in the time-domain.

We want to replace this model by the low-order model:

\[ \frac{d}{dt} \tilde{q}(t, \tilde{x}) + \tilde{j}(t, \tilde{x}) = \tilde{B}u, \quad \tilde{x}(0) = \tilde{x}_0 \]
\[ y = \tilde{D}\tilde{x} \]
Applications

- If the nonlinear behaviour of the original model is essential.
- Optimization loop needs fast but accurate models.
- Verification of IC designs only considers the input-output behaviour.
- Design of controllers need models of low dimension.
- Models with different time-scales
- Applications in electrical engineering, aeronautic engineering, mechanical engineering, etc.
Nonlinear balanced truncation I

- Observability and controllability functions
  \[ L_c(x_0) = \min \left\{ \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 dt : u \in L_2(-\infty, 0), x(-\infty) = 0 \right\}, \]
  \[ L_o(x_0) = \frac{1}{2} \int_{0}^{\infty} \|y(t)\|^2 dt, \forall \tau \in [0, \infty) u(\tau) = 0. \]

- These functions can be solved from PDEs of Lyapunov and Hamilton-Jacobi type.
- The transformation \( x = \phi(z) \) balances the system if
  \[ L_o(\phi(z)) = \frac{1}{2} z^T z \]
  \[ L_c(\phi(z)) = \frac{1}{2} z^T \text{diag}(\tau_1(z), \ldots, \tau_n(z)) z \]
  \[ \frac{\partial L_c}{\partial z_i}(\phi(z)) = 0 \iff \frac{\partial L_o}{\partial z_i}(\phi(z)) = 0 \]

where \( \tau_1(z) \geq \ldots \geq \tau_n(z) \) are the singular value functions.
Nonlinear balanced truncation II

- If \( \tau_n(\ldots, z_n) \geq 0 \) the axis singular value functions \( \rho_i \) can be defined by
  \[
  \tau_i(\ldots, z_i, \ldots) = \rho_i^2(z_i).
  \]

- The \textbf{Hankel norm} for nonlinear systems equals
  \[
  \|\Sigma\|^2_H = \max_{u \in L_2[0, \infty)} \frac{\|H(u)\|^2}{\|u\|^2} = \max_x \frac{L_o(x)}{L_c(x)} = \max_s \{\rho_1(s)\}.
  \]

- This balanced form can be used for nonlinear model reduction.
- Stability is preserved.
- Generalization of balanced truncation
Trajectory Piecewise Linear (TPWL)

- Nonlinear model is approximated by piecewise linear model along a trajectory;
- In each ball the linear model is reduced by linear MOR techniques, which results in a basis $P_i$;
- Calculate the SVD of $\tilde{P} = [P_1, \ldots, P_N]$ and define the global subspace as $P = U(:, 1: r)$.
- Create a weighted representation of the reduced model.
- Evaluation costs are also reduced!
Introduction

Trajectory Piecewise Linear (TPWL)

Proper Ortogonal Decomposition for DAEs

POD

- We need to compute snapshots from the nonlinear system.
- A SVD is made from the correlation matrix;
- Then a global basis $\Phi$ is derived;
- Project the problem:
  \[
  \frac{d}{dt} \Phi^T q(t, \Phi \tilde{x}) + \Phi^T j(t, \Phi \tilde{x}) = \Phi^T Bu, \quad \tilde{x}(0) = \tilde{x}_0
  \]
  \[
  y = D\Phi\tilde{x}
  \]
- Notice that the evaluation costs for POD are not reduced!
Empirical balanced truncation (EBT)

- The **empirical** observability and controllability Gramians are derived from several **snapshot series** with representative inputs and initial values.
- The **global basis** is derived by balancing these empirical Gramians.
- Same projection as POD.
- Input-output behaviour is taken into account;
- Very **expensive** model extraction
Volterra series

- The method is only applicable if the system depends linearly on the inputs.

\[ \dot{x} = f(x) + Bu(t). \]

- A bilinear system for

\[ x^\otimes = \begin{bmatrix} x \\ x \otimes x \\ \vdots \end{bmatrix} \]

is constructed which approximates the first moments of the nonlinear system.

\[ \frac{d}{dt} x^\otimes = A^\otimes x^\otimes + \sum_{i} N_i^\otimes u_i(t) x^\otimes + B^\otimes u(t). \]

- This bilinear system can be reduced by nested Krylov methods, e.g. Arnoldi.

The method is not applicable for DAEs because the pencil
Comparison of nonlinear MOR methods

- For **nonlinear DAEs** only TPWL, POD and EBT are possible.
- TPWL and EBT really match input-output behaviour, in contradiction to POD.
- POD and EBT need snapshots of the nonlinear system, which can be expensive.
- Only TPWL also reduces the model **evaluation costs**!
Some literature of nonlinear MOR

Antoulas, Sorensen: General overview of MOR
Scherpen, Fujimoto: Nonlinear balanced truncation
White, Rewienski, Voss: TPWL
Astrid, Wilcox, Petzold: POD
Lall, Marsden, Glavaski: EBT
Phillips, Bai, Schetzen: Volterra series
Optimization problem

- For all MOR techniques we are looking for a transformation $x = \phi(z)$, which balances the system or the input-output behaviour (Hankel operator).
- This means that for all $r$ the first $r$ elements of the transformed system is automatically the optimal solution, which minimizes the reduction error! This can be the error of the state or the input-output map.
- Such a balanced system can easily be reduced by selecting only the first $r$ equations and variables.
- For linear systems one get always the linear transformation $x = Pz$, where $P$ solves a system of Lyapunov equations and can be approximated by Krylov methods.
Introduction

Trajectory Piecewise Linear (TPWL)

Proper Orthogonal Decomposition for DAEs

Classification of MOR methods

- **Global** transformations do not depend on $t$, in contradiction to local transformations. They balance the system on the complete interval. For nonlinear systems they always need **snapshots** combined with a **SVD** in order to get a global basis.

- **Local** transformations balance the system along a **short time** interval on which the system matrices can be assumed to be constant! They do not need empirical data but apply linear methods to the **locally linearized** system.
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The system is linearized at some linearization tuples (LT) \((x_{l_i}, t_{l_i})\). In each region the linearized systems are reduced by linear model reduction techniques. The LT are selected at a trajectory of typical initial values and inputs. Afterwards a weighted sum is constructed of all reduced linearized systems.
TPWL sequence I

- Construct $s$ linear reduced systems along a trajectory
  \[ C_{l_i} \dot{x} + G_{l_i} x + B_{l_i} u(t) = 0. \]

- Reduce each system with a linear MOR technique to get $P_{l_i}$ which represents the reduced subspace of the $i$-th system.

- Calculate a SVD of $\tilde{P} = [P_{l_1}, \ldots, P_{l_s}] = USV^T$ and define the global subspace as $P := U(:, 1: r)$.

- Reduce each linearized system with $P$
  \[ P^T C_{l_i} P \dot{y} + P^T G_{l_i} P y + P^T B_{l_i} u(t) = 0. \]
TPWL sequence II

- Create a weighted representation

\[ \sum_{i=0}^{s-1} w_i(y, t) \left( C_{ir} \dot{y} + G_{ir} y + B_{ir} u(t) \right) = 0, \]

where \( C_{ir} = P^T C_{li} P, \ G_{ir} = P^T G_{li} P, \ B_{ir} = P^T B_{li} \) and \( w_i(y, t) \) is a weighting function which describes how much influence the \( i \)-th reduced linearized system has on the actual state.
Introduction

Trajectory Piecewise Linear (TPWL)

Proper Orthogonal Decomposition for DAEs

Linear model order reduction

Krylov  These methods match a certain number of moments of the transfer function. For this type it seems that PRIMA is the best one because of preserving stability and passivity. All Krylov based methods are very fast but create too large subspaces.

TBR  This method balances the system first and then truncate the modes which are hardly to observe or to control. In theory this method should perform very well but in practice there are a lot of problems in the case of DAE models.

Poor Mans TBR  This is a combination of multi point rational interpolation and TBR. Therefore the transfer function is calculated at several frequencies and then a SVD is used to find an optimal subspace. This method is slower than Krylov but creates better approximations.
Linearization tuple controller

- Error of TPWL model consists of *linearization* and *reduction* error.
- Locations of linearization tuples determine the linearization error.
- These tuples can be created during a *rough* transient simulation, but with the restriction that this rough solution lies in the neighbourhood of the exact solution.
Weighting

- Weighting is needed to combine the local reduced systems to a global linear system.
- Distance \( d_i = \| y - y_{li} \| \) and minimum distance \( m \).
- Weighting function:

\[
    w_i(y, t) = \frac{1}{S} e^{-\frac{\gamma d_i}{m}}.
\]

- Extended weighting using the Hessians of \( q \) and \( j \).

\[
    d_i = \frac{1}{2} H_{j,i} \| Py - x_i \|^2 + H_{q,i} \| Py - x_i \| \| P\dot{y} \|,
\]

where

\[
    H_{j,i} \geq \left\| \frac{\partial^2}{\partial x^2} j(t, x) \right\|, \quad H_{q,i} \geq \left\| \frac{\partial^2}{\partial x^2} q(t, x) \right\|.
\]
### Inverter chain model

- The state of the inverter chain model comprises of 100 nodal voltages and some electrical currents. In total, there are 104 state variables.
- The excitation signal $U_{op}$ and the dynamical response of the inverter chain.

![Diagram of inverter chain model](chart.png)
Numerical results of TPWL

- Proper numerical results for $r = 50$.
- The higher accuracy of PMTBR can be used to get smaller models.
- Reduced models rather already still valid for different inputs.
- Linearization tuples are directly computed during a rough transient simulation before.
- Weighting using the norms of the Hessians gives the best results.
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- POD constructs a subspace which should consist the most dominant part of the state of the nonlinear system.
- In contradiction to TPWL and EBT the state itself is approximated.
- Snapshots are needed to create the reduced basis.
- In literature POD is only applied to ODEs.
- POD could also be applied in a trajectory piecewise approach like TPWL.
**POD procedure**

- Collection of simulation ($N$ snapshots)

\[ T_{\text{snap}} = [t(1), \ldots, t(N)] . \]

- Construction of correlation matrix \( M = \frac{1}{N} T_{\text{snap}}^T T_{\text{snap}} \).

- Eigenvalue decomposition of correlation matrix:

\[ M = U \Sigma U^T . \]

- *Galerkin* projection by \( \Phi = U(:, 1 : r) \):

\[
\frac{d}{dt} \Phi^T q(t, \Phi \tilde{x}) + \Phi^T j(t, \Phi \tilde{x}) = \Phi^T B u, \quad \tilde{x}(0) = \tilde{x}_0 \\
y = D \Phi \tilde{x}
\]
Problems of POD

- Reduced **DAEs** are not always **solvable** in contradiction to ODEs.
- Function **evaluation costs** are not reduced at all.
Least squares approach

Instead of solving \( \frac{d}{dt} \Phi^T q(t, \Phi \ddot{x}) + \Phi^T j(t, \Phi \ddot{x}) = \Phi^T Bu \) we consider the problem:

\[
\frac{d}{dt} q(t, \Phi \ddot{x}) + j(t, \Phi \ddot{x}) = 0.
\] (1)

The (weak) solution \( \ddot{x} \) should minimize the residual of this equation.

Least squares system

\[
\Phi_n^T M^T M \Phi_n y = \Phi_n^T M^T b.
\]

Now the reduced DAE for POD is always solvable.
Missing Point Estimation

- Function evaluation $\Phi_n^T f(\Phi_n \tilde{x})$ is cheap if $\Phi_n$ is a permutation matrix.
- Let $P \in \{0, 1\}^{G \times K}$ be a permutation matrix where $G << K$ and define the restricted basis $\tilde{\Phi}_n$ as

$$\tilde{\Phi}_n = P \Phi_n. \quad (2)$$

- Combinatorial optimization problem

$$\text{find } P \quad \text{such that } \| (\Phi_n^T P^T P \Phi_n)^{-1} - I_n \| < \text{TOL} \quad (3)$$

subject to

$$PP^T = I$$
$$p_{ij} \in \{0, 1\}$$

- Approximation: $\text{cond}(\Phi_n^T P^T P \Phi_n) < \text{TOL}$. 

Numerical results of POD

The POD basis $\Phi$ is found by solving the eigenvalue problem for the correlation matrix.

The POD basis $\Phi$ comprises the eigenvectors corresponding to 20 largest eigenvalues.
Comparison of TPWL and POD

- For the same basis the TPWL model is much cheaper to evaluate than POD because all local Jacobians are already reduced and stored. This is also the case if MPE is used.
- Therefore TPWL needs much more storage than POD.
- For very nonlinear models TPWL needs a lot of linearization tuples, which implies a large reduced basis.
- In general the POD basis can be of smaller size, because POD directly projects the nonlinear model and only uses the data of one simulation.
- TPWL will be more robust with respect to changes in the parameters and inputs.