A general compound multirate method for circuit simulation problems

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Introduction

Electrical circuits are modeled by the following differential-algebraic equations.

$$\frac{d}{dt}\left[\mathbf{q}(t,\mathbf{x})\right] + \mathbf{j}(t,\mathbf{x}) = 0.$$

The functions $\mathbf{q}, \mathbf{j}$ represent the charges on capacitors and currents through resistors and sources in the circuit, while the state vector $\mathbf{x}$ consists of voltages and currents.

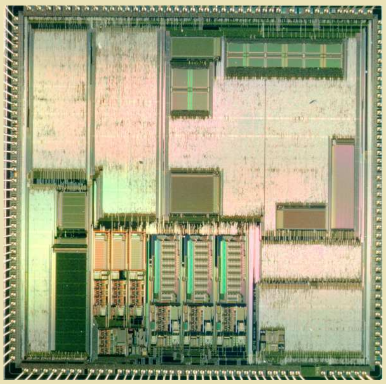

Figure 1. *A mixed digital-analog circuit.*

Often, parts of electrical circuits have latency or multirate behaviour. Latency means that parts of the circuit are constant during a certain time interval.

Multirate methods for circuits

Assume that the model of the circuit can be partitioned into an active (A) and a latent (L) part.

$$\begin{cases} \frac{d}{dt}\left[\mathbf{q}_A(t,\mathbf{x}_A,\mathbf{x}_L)\right] + \mathbf{j}_A(t,\mathbf{x}_A,\mathbf{x}_L) = 0 \\ \frac{d}{dt}\left[\mathbf{q}_L(t,\mathbf{x}_A,\mathbf{x}_L)\right] + \mathbf{j}_L(t,\mathbf{x}_A,\mathbf{x}_L) = 0 \end{cases}$$

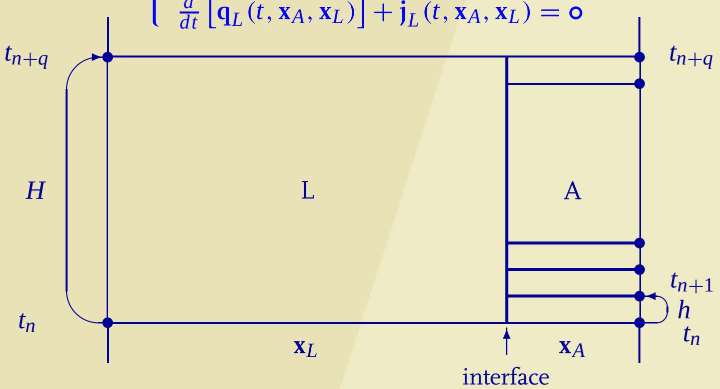

Figure 2. *Schematic overview of a multirate step.*

Figure 2 shows a schematic overview of one step of a "Slowest First" (SF) multirate method. First the latent part $\mathbf{x}_L$ is computed at $t_{n+q} = t_n + H$, while $\mathbf{x}_A$ is extrapolated. Then the active part is integrated at a refined grid with a much smaller stepsize $h = \frac{H}{q}$, while $\mathbf{x}_L$ is approximated by linear interpolation.

The "General Compound" (GC(α)) methods use implicit extrapolation during the first part. Simultaneously with x_L at t_{n+q}, x_A at $t_n + \alpha H$ is calculated ($\alpha = \frac{1}{q}$, or $\alpha = 1$).

Stability analysis

For multirate methods it is not possible to analyse the stability on the scalar test equation. Therefore we have studied the following (real) linear test equation.

$$\begin{pmatrix} \dot{x}_A \\ \dot{x}_L \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x_A \\ x_L \end{pmatrix}.$$

It can be shown that the SF and GC(α) methods are conditionally stable for $H \to 0$ if the matrix $\mathbf{A}$ is stable. We have also derived sufficient stability conditions for $H > 0$ and $q \to \infty$, which are shown in Table 1.

Table 1. *Sufficient stability conditions.*

SF	GC(α)
$a_{11} < 0$	$a_{11} < 0$
$a_{22} < 0$	$\alpha a_{11} + a_{22} < 0$
$\lvert a_{12}a_{21}\rvert < \lvert a_{11}a_{22}\rvert$	$-a_{11}a_{22} - 2\alpha a_{11}^2 < a_{12}a_{21} < a_{11}a_{22}$

Clearly the third condition for GC(α) is satisfied if

$$\lvert a_{12}a_{21}\rvert < \lvert a_{11}a_{22}\rvert.$$

Test problem

Consider for $0 \geq t \geq 10$

$$\begin{pmatrix} \dot{x}_A \\ \dot{x}_L \end{pmatrix} = \begin{pmatrix} -1 & \mu \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_A \\ x_L \end{pmatrix}, \quad \begin{pmatrix} x_A(0) \\ x_L(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Sufficient stability conditions are $\lvert\mu\rvert < 1$ or $-1 - 2\alpha < \mu < 1$ for the GC(α) methods.

Table 2. *Stability of multirate methods ($H = 0.1, q = 10$).*

μ	SF	GC($\frac{1}{q}$)	GC(1)
−10	√	√	√
−100	-	√	√
−1000	-	-	√

Future developments

The results will be generalized to the general multi-dimensional case.

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