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Condition number of the boundary element method matrices

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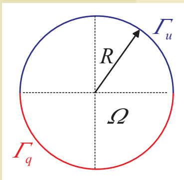
Introduction

The boundary element method (BEM) transforms a partial differential equation into a boundary integral equation, and, after discretisation of the boundary of the domain, into a set of linear algebraic equations. The matrix of this set of equations has a special structure, and we investigate the condition number of the matrix.

Equations

We focus on the Laplace problem on a circular domain with mixed boundary conditions.



Partial differential equation (PDE)

$$\begin{cases} \Delta u &= 0, & \boldsymbol{x} \in \Omega, \\ u &= \tilde{u}, & \boldsymbol{x} \in \Gamma_u, \\ q &:= \frac{\partial u}{\partial n} = \tilde{q}, & \boldsymbol{x} \in \Gamma_q, \end{cases}$$

We seek for u and q on the boundary, which are partly given by the boundary conditions.

$$\partial\Omega = \Gamma = \Gamma_1 \bigcup \Gamma_2.$$

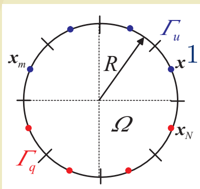
Boundary integral equation

Using the fundamental solution for the Laplace operator and Green's second identity, we transform the PDE into a boundary integral equation.

$$\mathcal{K}^s q = \left(\frac{1}{2}\mathcal{I} + \mathcal{K}^d\right)u.$$

Algebraic equations

The boundary Γ is divided into N linear elements Γ_k; in each element we have a nodal point $\boldsymbol{x}_k$. The first m elements have Dirichlet conditions; the last $N-m$ elements have Neuman conditions.



Discretisation

The boundary integral equation transforms in a set of linear algebraic equations,

$$\mathbf{G}\boldsymbol{q} = \left(\frac{1}{2}\mathbf{I} + \mathbf{H}\right)\boldsymbol{u} =: \tilde{\mathbf{H}}\boldsymbol{u},$$

with the elements of the matrices $\mathbf{H}$ and $\mathbf{G}$ given. We rearrange the columns in such a way that all unknowns appear at the left-hand side,

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{f}, \quad \mathbf{A} = \left[-\mathbf{G}\mathbf{P}_1 \mid \tilde{\mathbf{H}}\mathbf{P}_2\right].$$

Here, $\mathbf{P}_1$ and $\mathbf{P}_2$ are projection matrices,

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{I}_m \\ \hline \emptyset \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} \emptyset \\ \hline \mathbf{I}_{N-m} \end{bmatrix}. \qquad (1)$$

Extreme cases

In the extreme cases that $m = N$ or $m = 0$, we are dealing with the Dirichlet and Neuman problem, respectively. For those problems, we can get the condition number explicitly.

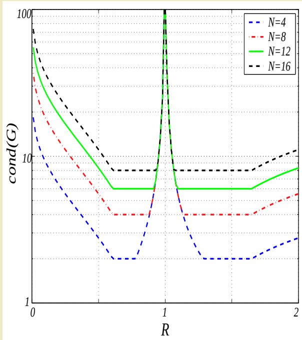
Dirichlet problem $(m = N)$

$$\mathrm{cond}(\mathbf{A}) = \frac{\max\left(\frac{1}{2}, |\log R|\right)}{\min\left(\frac{1}{N}, |\log R|\right)}.$$

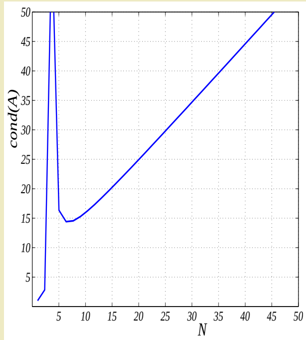
Neuman problem $(m = 0)$

$$\mathrm{cond}(\mathbf{A}) = \frac{\pi + \tan\frac{\pi}{N}}{|\pi - (N-1)\tan\frac{\pi}{N}|}.$$

In the figures below, we show the behavior of the condition numbers. In the Dicirchlet case, $\mathrm{cond}(\mathbf{A})$ is singular for $R = 1$. For the Neuman case, the condition number is independent of R.



Dirichlet problem



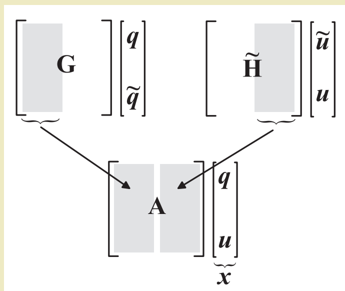
Neuman problem

Mixed boundary conditions

The matrix $\mathbf{A}$ is constructed from the first $(N-m)$ columns of $\mathbf{G}$ and the last m columns of $\tilde{\mathbf{H}}$. We decompose $\mathbf{A}$ into

$$\mathbf{A} = \left[\mathbf{Q}_1 \mid \mathbf{Q}_2\right] \begin{bmatrix} \mathbf{U}_1 & \emptyset \\ \emptyset & \mathbf{U}_2 \end{bmatrix}.$$

The matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are orthogonal, and $\mathbf{U}_1$ and $\mathbf{U}_2$ are upper triangular.



$\mathbf{A}$ is constructed from $\mathbf{G}$ and $\tilde{\mathbf{H}}$

The idea is that the matrices $\mathbf{Q}_i$ and $\mathbf{U}_i$ are well-conditioned, and the angle between $\mathbf{Q}_1$ and $\mathbf{Q}_2$ small. Therefore, the condition number of $\mathbf{A}$ should be small.

Future work

We want to determine the condition number for the case of mixed boundary conditions. We hope to establish this by using the mentioned decomposition.