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Mixed Finite Element for Swelling of Cartilaginous Tissues, I

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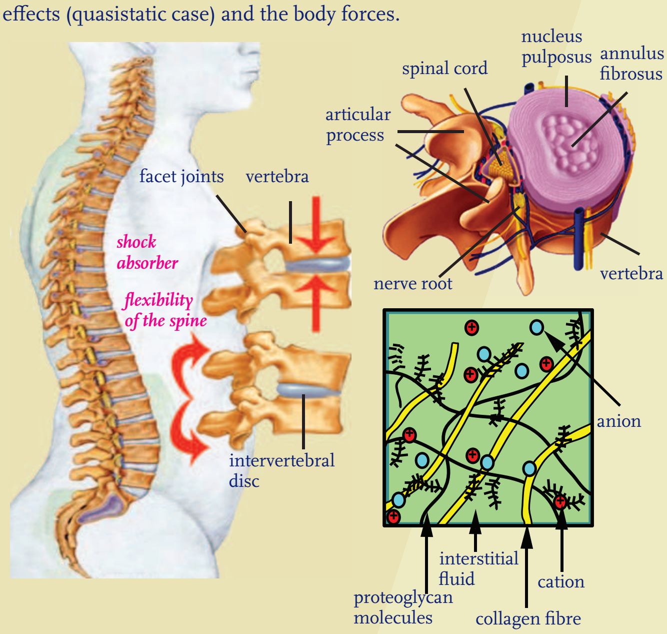
Introduction

The swelling and shrinking of cartilaginous tissues (CT) (like Intervertebral Disks) can be modelled by a four-component mixture theory in which a deformable and charged porous medium is saturated with a fluid with dissolved ions. This theory results in a coupled system of non-linear parabolic differential equations together with an algebraic constraint for electro-neutrality.

Four-Component Modeling of CT

- The two-component (Biot classical model) mixture theory is not able to describe the swelling and shrinking, that is caused by chemical and/or electrical loads.
- Therefore the theory is extended to four-component mixture theory.
- We assume incompressible and linear elastic solid saturated with incompressible and Newtonian viscous fluid. Also we neglect the inertial effects (quasistatic case) and the body forces.



Balance Equations

(degrees of freedom, material parameters)

$$\begin{cases} -\nabla\cdot(2\mu_s\boldsymbol{\mathcal{E}}(\mathbf{u})+\lambda_s\nabla\cdot\mathbf{u})+\nabla p=\mathbf{0}, & \text{force balance}\\ \dfrac{\partial\nabla\cdot\mathbf{u}}{\partial t}+\nabla\cdot\mathbf{q}^l=0, & \text{volume balance}\\ \dfrac{\partial c^\beta(\nabla\cdot\mathbf{u}+\varphi_0)}{\partial t}+\nabla\cdot(\mathbf{q}^\beta+c^\beta\mathbf{q}^l)=0,\ \beta=+,-. & \text{ion balances}\end{cases}$$

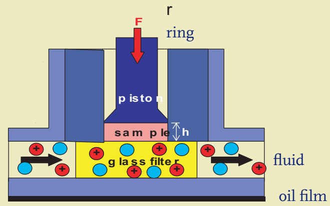
Constitutive Equations

$$\begin{cases} \mathbf{q}^l=-K(\nabla\mu^l+c^+\nabla\mu^++c^-\nabla\mu^-), & \text{Darcy's law}\\ \mathbf{q}^\beta=-\dfrac{D^\beta c^\beta\varphi}{RT}\nabla\mu^\beta,\quad \beta=+,-. & \text{Fick's law}\end{cases}$$

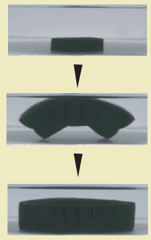
Secondary Equations

$$\begin{aligned} \varphi &= 1-(1-\varphi_0)(1-\nabla\cdot\mathbf{u}),\quad c^{fc}=c_0^{fc}(1-\nabla\cdot\mathbf{u}/\varphi_0),\\ c^\beta &= \frac{1}{2z^\beta}c^{fc}+\frac{1}{2}\sqrt{(c^{fc})^2+4c^2exp\frac{(\mu^+-\mu_0^++\mu^--\mu_0^-)}{RT}},\\ p &= \mu^f-\mu_0^l+RT(\Gamma^+c^++\Gamma^-c^-),\\ \xi &= \frac{1}{z^\beta F}(\mu^\beta-\mu_0^\beta-RT\ln\frac{c^\beta}{c}),\qquad \beta=+,-. \end{aligned}$$

Experiments and Boundary Conditions



Experimental Set-Up



Experiment on hydrogel disc
J.M. Huyghe 1999

Mixed Finite Element

- For the sake of ***local mass conservation***, equations are discretised in space using mixed finite element.
- We choose $\mathbf{u}\in\mathcal{P}_1(\Omega)^2$, $\mathbf{q}^l$, $\mathbf{q}^\beta_{tot}:=\mathbf{q}^\beta+c^\beta\mathbf{q}^l\in\mathcal{RT}_0(\Omega)$ and μ^l, $\mu^\beta\in\mathcal{P}_0(\Omega)$.
- Implicit time integration results into a non-linear set of equations.
- Since different scales are apparent in the weak formulation, then we replace the reduced matrices by the scaled ones.
- Substitution in the discrete variational formulation gives:

$$\boxed{\mathfrak{A}(\varphi,c^+,c^-)\frac{d\mathbf{y}}{dt}+\mathfrak{B}(\varphi,c^+,c^-)\mathbf{y}=\mathfrak{F}(\varphi_h,c_h^+,c_h^-)+\frac{d\mathfrak{G}}{dt}(\varphi_h,c_h^+,c_h^-),}$$

$$\mathbf{y}=[\tilde{\mathbf{u}},\tilde{\mathbf{q}}^l,\tilde{\mathbf{q}}^+_{tot},\tilde{\mathbf{q}}^-_{tot},\tilde{\boldsymbol{\mu}}^l,\tilde{\boldsymbol{\mu}}^+,\tilde{\boldsymbol{\mu}}^-]^T,$$

$$\mathfrak{A}_{ij}=\begin{cases}\mathbf{B}^T, & i=5,\ j=1,\\ \mathbf{0}, & i\neq 5,\ j\neq 1,\end{cases}$$

$$\mathfrak{B}=\begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}^{ll}(\varphi_h/c_h^\beta) & \mathbf{C}^{l+}(\varphi_h) & \mathbf{C}^{l-}(\varphi_h) & \mathbf{D} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}^{l+}(\varphi_h) & \mathbf{C}^{++}(\varphi_h c_h^+) & \mathbf{0} & \mathbf{0} & \mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}^{l-}(\varphi_h) & \mathbf{0} & \mathbf{C}^{--}(\varphi_h c_h^-) & \mathbf{0} & \mathbf{0} & \mathbf{D}\\ \mathbf{0} & \mathbf{D}^T & \mathbf{0} & \mathbf{0} & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{D}^T & \mathbf{0} & & \mathbf{O}_{3K\times 3K} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}^T & & & \end{pmatrix}$$

- The upper-left block is symmetric positive definite.
- Given φ_h and c_h^β, the solution for the linear saddle point problem exists and it is unique.

Analytical Solution

- To verify the numerical solutions we derived a set of analytical solutions for the one-dimensional problem.
- The ide behind is to derive a coupled system of diffusion equations.

$$\boxed{\frac{\partial}{\partial t}\begin{pmatrix}\mu^l\\ \mu^+\\ \mu^-\end{pmatrix}=\mathbf{EP}\frac{\partial^2}{\partial x^2}\begin{pmatrix}\mu^l\\ \mu^+\\ \mu^-\end{pmatrix}}$$

- $\mathbf{E}$ and $\mathbf{P}$ are both symmetric positive definite and related to the Hessian of Helmholtz free energy and the diffusion matrix, respectively.

References

- Malakpoor et al., An analytical solution of incompressible charged porous media , ZAMM. Z. Angew. Math. Mech, 2006.
- Malakpoor et al., Mathematical modelling and numerical solution of swelling of cartilaginous tissues. Part I. ESAIM: Mathematical Modelling and Numerical Analysis, 2006.

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