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Stability and evolution of gravity-driven porous media flows

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Introduction

Upflowing salty groundwater, evaporating completely at the ground surface, leads to the buildup of a **saline boundary layer** below the surface. This boundary layer, if stable, may grow to a finite thickness at equilibrium.
Thus we have a fluid layer that **differs in density** from the fluid below, and the question of the **gravitational stability** of this boundary layer arises. From numerical and Hele–Shaw cell experiments, it appears that the saline boundary layer may be unstable to perturbations, resulting in so called **"salt-fingers"**.

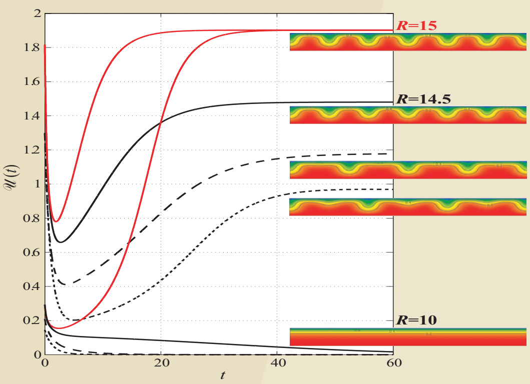
Dimensionless model equations

- Transport equation . . . . $\dfrac{\partial S}{\partial t} + \mathrm{div}(R\,\mathbf{U}S - \mathrm{grad}\,S) = 0$,
- Darcy law . . . . $\mathbf{U} + \mathrm{grad}\,P - S\mathbf{e}_z = \mathbf{0}$,
- Fluid incompressibility . . . . $\mathrm{div}\,\mathbf{U} = 0$,

subject to suitable boundary- and initial conditions.
The bifurcation parameter R is usually referred to as the system **Rayleigh number**. The **ground state** solution is characterized by the uniform (constant) flow $\mathbf{U}_0$ and the equilibrium one-dimensional boundary layer $S_0(z)$.

Questions: Under what conditions is the ground state stable/unstable, and what happens when it becomes unstable ?

Growing instabilities and pattern formation



FEM simulations of the full model. Above a certain critical Rayleigh number the system bifurcates to a convective regime. The shape and number of fingers depend on the initial conditions.

Reduction to a lower-order dynamical system

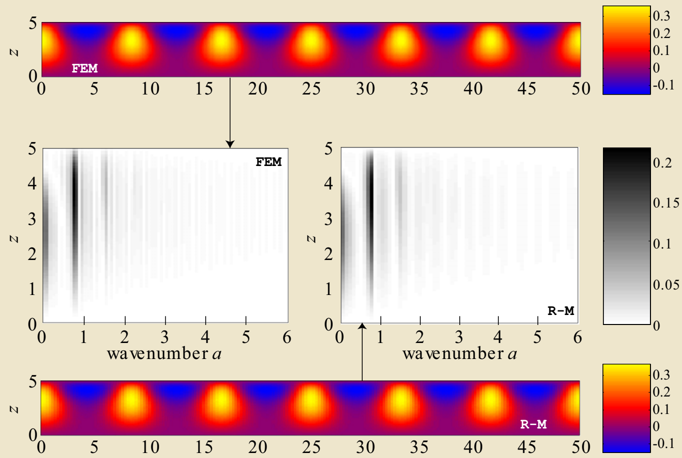
Claim: dynamics of the fully nonlinear problem can be captured by a lower-order model. Idea:

- prescribe the patterns (e.g. rolls, hexagons, ...) in the (x, y)-plane via planform functions having certain (dynamically active) wavenumbers;
- use eigenfunctions of the linearised problem for the expansion in the z-direction;
- Galerkin–Eckhaus method: project the fully nonlinear model onto this pattern expansion.

This gives the system of nonlinear ODE's:

$$a_j \frac{\mathrm{d}A_j(t)}{\mathrm{d}t} = \sum_{k=1}^{N} b_{jk} A_k(t) + \sum_{k=1}^{N}\sum_{l=1}^{N} c_{jkl} A_k(t) A_l(t) \; ,$$

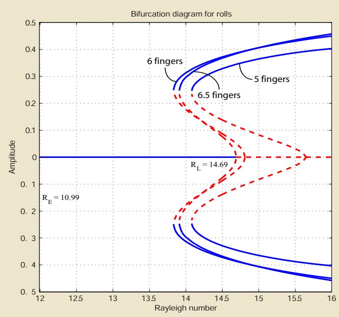
where A_j denotes the amplitude of the j-th mode in the pattern expansion. This equation is sometimes called the **Landau equation** or amplitude equation.
The dynamically active eigenmodes are not *a priori* known. However, using Fourier analysis one obtains detailed information about these active modes.

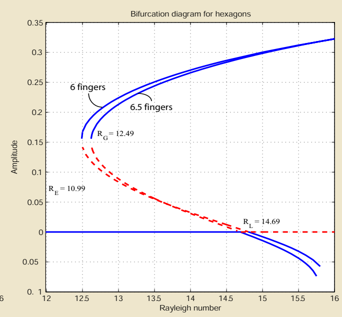


(Top figure) 2D FEM numerical simulation of a perturbation pattern. The three active wavenumbers in the FEM simulations are used for the order reduction. As to be expected, only these three horizontal modes plus five eigenfunctions in z-direction represent the complete pattern (bottom figure). The system Rayleigh number for both cases is given by $R = 14.5$.

Benefit reduced model: Bifurcation analysis with the reduced model is from a numerical point of view cheaper (3D patterns !) and gives more information about the stability of the steady-states:

- Roll patterns
- Hexagonal patterns





Blue depicts stable branches, the dashed red ones unstable. The rolls bifurcate subcritically while the hexagonal patterns bifurcate transcritically. Pattern interaction will probably lead to a combination of both bifurcation diagrams.

Future developments

- Branch following for larger Rayleigh numbers
- Pattern interaction: rolls with hexagons, squares etc.
- Stochastic determination of the attraction basins

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