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Numerical Methods for Solving Flow of ODE

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Problem Setting

For an autonomous problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x}(0)$ is any point in an interval $\mathbf{I}(0)$ say, the solution $\mathbf{x}$ is called *a flow*.

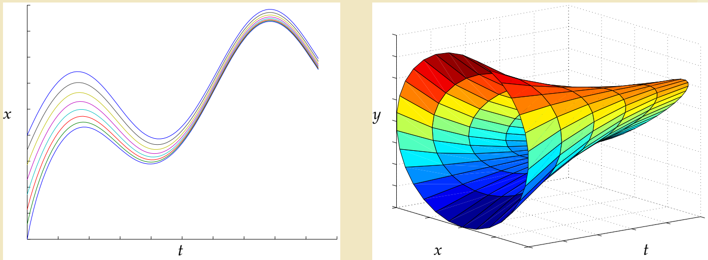

Figure 1: Flow examples in 1D and 2D

For stiff problems one needs implicit method to find the solution without stability constraints. In this project we show how we can exploit the autonomous character of the problem and derive explicit methods having the same favourable stability properties.

Methods

The so-called *flow methods*, for obtaining the numerical solution, has several main aspects:

Time discretisation. A set of time levels t^i which spans the integration time $[0, t]$ is needed.

Space discretisation. It should be such that enough information about the shape of $\mathbf{I}(t)$ and about the vector field is provided. The set of discretisation points $\mathbf{x}_k$ may consist of only points of the solution, but it can also be extended to contain additional space points.

Implicit numerical method. This is of interest because of possible stiffness. **Example**: Euler Backward (EB)

$$\mathbf{x}_k^{i+1} - \Delta t\, \mathbf{f}(\mathbf{x}_k^{i+1}) = \mathbf{x}_k^i.$$

In general, flow methods can be based on the following methods:

- ILM (Implicit Linear Multistep) methods,
- DIRK (Diagonally Implicit Runge-Kutta) methods.

Autonomous vector field. It allows us to approximately find $\mathbf{f}(\mathbf{x}_k^{i+1})$ based on information at previous time steps or time stages.

Inverse interpolation. We employ the approximation $\mathbf{P}(\mathbf{x})$ of the inverse function of $\mathbf{F}(\mathbf{x}) = \mathbf{x} - \Delta t\, \mathbf{f}(\mathbf{x})$ present in all implicit numerical methods (listed above) to obtain the solution at the next time-level $\mathbf{x}_k^{i+1} = \mathbf{P}(\mathbf{x}_k^i)$.

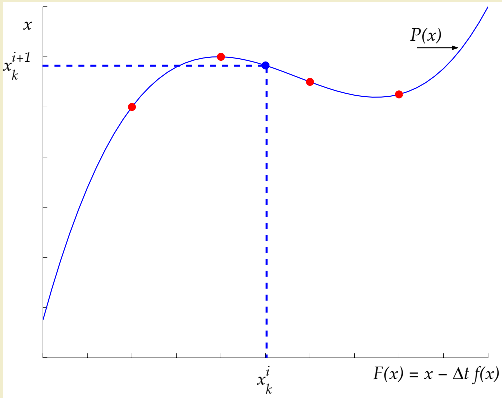

Figure 2: Inverse interpolation (1D) in the flow methods

Results

The accuracy of the flow methods is determined by the accuracy of the original (implicit) methods and the interpolation error.

Example 1 A simple scalar nonlinear problem $\dot{x} = -x^3$, $x(0) \in [-1, 1]$, solved by the flow methods based on BDF (FBDF*i*) of various order i. Table 1 shows the values in ∞-norm of the global error (at $t = 1.0$, with $\Delta t = 0.1$).

	FBDF1	FBDF2	FBDF4	FBDF6
$\|\|e\|\|_\infty$	1.5e-02	2.9e-03	2.8e-04	5.3e-05

Table 1: Accuracy of the flow methods based on BDF methods

Example 2 The flow methods can also be used for higher dimensional problems. The interpolation is then also higher dimensional. Figure 3 shows the evolution in time of the half-ellipse, under the influence of the velocity field $\mathbf{f}(x, y) = [-\frac{1}{2}x^2 \cos y,\ \ x \sin y]^T$. The flow method used is based on the implicit midpoint rule.

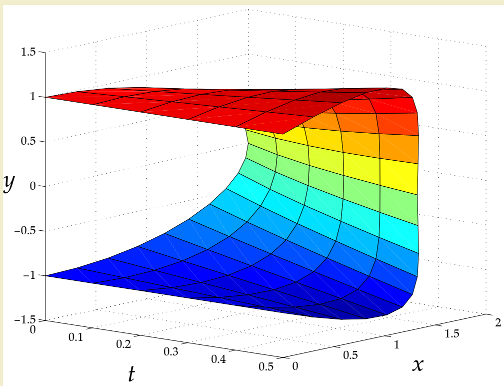

Figure 3: Evolution in time of the half-ellipse