A Note on Convective Terms in the Dual Reciprocity Boundary Element Method

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Abstract

In this note, we present an easier approach to deal with the first order derivatives of the problem variable in the context of the dual reciprocity boundary element method. This new approach saves computational work and improves accuracy. Examples are given to show its advantages.

1 Introduction

To extend the scope of application of the boundary element method (BEM), the dual reciprocity method (DRM) has been playing its important role. Besides inhomogeneous problems, the DRM has been successfully used to non–linear problems, time–dependent problems, anisotropical problems and so on so forth. Many engineering problems involve the first order derivatives of the problem variable, to be precise, we consider the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}. \tag{1}$$

As we can see, the equation (1) is of Poisson type, be the right hand side is unknown. Logically, let's start with a Poisson equation with a given right hand side

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = b(x, y). \tag{2}$$

To solve equation (2), the right hand side b is first expanded in terms of a radial basis function (RBF) (Powell[2]) $\phi(r)$ (by interoplation for instance)

$$b = \sum_j \alpha_j \phi(r_j). \tag{3}$$

where $r_j = \|\mathbf{x} - \mathbf{x}_j\|_2$, $\mathbf{x} = (x, y)$, $\mathbf{x}_j = (x_j, y_j)$, and $\mathbf{x}_j$ is the j-th RBF interpolation points. For $\phi(r_j)$, it normally easy to find a particular solution $\hat{u}_j$ such that (see Table 1)

$$\frac{\partial^2 \hat{u}_j}{\partial x^2} + \frac{\partial^2 \hat{u}_j}{\partial y^2} = \phi(r_j). \tag{4}$$

It means that

$$\hat{u} = \sum_j \alpha_j \hat{u}_j. \tag{5}$$

is an approximate particular solution to equation (2), i.e., $u - \hat{u}$ satisfies the Laplace equation

$$\frac{\partial^2 (u - \hat{u})}{\partial x^2} + \frac{\partial^2 (u - \hat{u})}{\partial y^2} = 0. \tag{6}$$

Implementing the standard BEM we have

$$\mathbf{H}(\mathbf{u} - \hat{\mathbf{u}}) - \mathbf{G}(\mathbf{q} - \hat{\mathbf{q}}) = 0. \tag{7}$$

where $\mathbf{u}$ is a vector of values of u at boundary points **and internal points**, $\mathbf{q}$ is a vector of normal derivatives at boundary points. $\hat{\mathbf{u}}$ and $\hat{\mathbf{q}}$ have the similar meaning.

By using express (5) we arrive at

$$\mathbf{Hu} - \mathbf{Qq} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\alpha. \tag{8}$$

where matrix $\hat{U}_{ij} = \hat{u}_j(\mathbf{x}_i)$, $\hat{Q}_{ij} = \frac{\partial \hat{u}_j}{\partial n}|_{\mathbf{x}=\mathbf{x}_j}$. α can be determined by equation (3), i.e.,

$$\mathbf{F}\alpha = \mathbf{b}. \tag{9}$$

where interpolation matrix $F_{ij} = \phi_j(\mathbf{x}_i)$, $\mathbf{b}$ is a vector of values of b. Therefore equation (8) becomes

$$\mathbf{Hu} - \mathbf{Qq} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\mathbf{F}^{-1}\mathbf{b}. \tag{10}$$

we bear equation (10) in mind, it is the pith and marrow of the DRM.

Now we return to our problem (1), we obviously have

$$\mathbf{Hu} - \mathbf{Qq} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\mathbf{F}^{-1}\frac{\partial \mathbf{u}}{\partial x}. \tag{11}$$

where notation $\frac{\partial \mathbf{u}}{\partial x}$ is a vector of $\frac{\partial u}{\partial x}$. To avoid new variables, a bridge should be established to connect $\mathbf{u}$ and $\frac{\partial \mathbf{u}}{\partial x}$. But by differentiating both sides of

$$u = \sum_j \alpha_j \phi(r_j). \tag{12}$$

we get

$$\frac{\partial \mathbf{u}}{\partial x} = \frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1}\mathbf{u}. \tag{13}$$

where matrix $(\frac{\partial F}{\partial x})_{ij} = \frac{\partial \phi(r_j)}{\partial x}|_{\mathbf{x}_i}$. Associating with equation (11)

$$\mathbf{Hu} - \mathbf{Qq} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1}\mathbf{u}. \tag{14}$$

Matrix equation (14) can be solved by applying boundary conditions.

names	RBFs	particular solutions
linear	r	$r^3/9$
cubic	r^3	$r^5/25$
thin plate spline	$r^2 \ln(r)$	$r^4 \ln(r)/16 - r^4/32$
Gaussian	$\exp(-r^2)$	$Ei(1, r^2)/4$
multiquadric	$\sqrt{c^2 + r^2}$	$\sqrt{c^2 + r^2}(4c^2 + r^2)/9$ $-c^3 \arctan(\sqrt{c^2 + r^2}/c)/3$
inverse multiquadric	$1/\sqrt{c^2 + r^2}$	$\sqrt{c^2 + r^2} - c \arctan(\sqrt{c^2 + r^2}/c)$

Table 1: RBFs and their particular solutions (1) (2D)

2 A new approach

The main idea of the DRM si to expand the right hand side to find an approximate particular solution. If the right hand side is unknown, for instance $\frac{\partial u}{\partial x}$, we may have an easier way to do this and do not introduce new variables. From expansion (12)

$$\frac{\partial u}{\partial x} = \sum_j \alpha_j \frac{\partial \phi(r_j)}{\partial x}. \tag{15}$$

The problem now is to find a particular solution $\bar{u}_j$, such that

$$\frac{\partial^2 \bar{u}_j}{\partial x^2} + \frac{\partial^2 \bar{u}_j}{\partial y^2} = \frac{\partial \phi(r_j)}{\partial x}. \tag{16}$$

It is quite easy to verify (althought it is difficult to find at first) that equation (16) has a particular solution which is shown in Table 2 for different RBF $\phi(r)$. We trace the same route as in section 1 and get a similar matrix equation

$$\mathbf{Hu} - \mathbf{Qq} = (\mathbf{H\bar{U}} - \mathbf{G\bar{Q}})\mathbf{F}^{-1}\mathbf{u}. \tag{17}$$

where $\bar{U}_{ij} = \bar{u}_j(\mathbf{x}_i)$, $\bar{Q}_{ij} = \frac{\partial \bar{u}_j}{\partial n}|_{\mathbf{x}=\mathbf{x}_j}$.

Comparing (14) with (17), we find (17) is easier and saves two times matrix multiplications. The new procedure doesn't approximate $\frac{\partial u}{\partial x}$ by RBF series, therefore it may more accurate.

The idea above can be applied to a right hand side which is a linear combination of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and u itself.

names	RBFs	particular solutions
linear	r	$xr/3$
cubic	r^3	$xr^3/5$
thin plate spline	$r^2 \ln(r)$	$xr^2(4\ln(r) - 1)/16$
Gaussian	$\exp(-r^2)$	$-x\exp(-r^2)/(2r^2)$
multiquadric	$\sqrt{c^2 + r^2}$	$x(c^2 + r^2)\sqrt{c^2 + r^2}/(3r^2)$
inverse multiquadric	$1/\sqrt{c^2 + r^2}$	$x\sqrt{c^2 + r^2}/r^2$

Table 2: RBFs and their particular solutions (2) (2D)

3 Numerical example

In this section we give an example which had been investigated in Partidge[1].

References

[1] P.W. Partridge, C.A. Brebbia & L.C. Wrobel, *The Dual Reciprocity Boundary Element Method*, 1992.

[2] M.J.D. Powell, *The theory of radial basis function approximation in 1990*, Advances in numerical analysis, Vol.II, 1992.