

# Alternative DRM formulations

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## Abstract

In this paper, we present two DRM formulations. For a pseudo–Poisson equation, if the right hand side is a linear operation on the dependent variable, we can derive a new DRM formulation. In comparison with the traditional DRM formulation for the same equation, the new one is much easier and more efficient. For the axisymmetric Poisson equation, we construct a DRM formulation by using the linear axisymmetric radial basis function. The particular solution involved is written in a closed form, and thus speeds up the evaluation of the particular solution. A few numerical examples demonstrate the accuracy and efficiency of these formulations.

## 1 Introduction

The boundary element method (BEM) is now a powerful alternative numerical technique for solving partial differential equations (PDEs). For homogeneous PDEs, only the boundary discretisation is necessary. For inhomogeneous PDEs, however, the integral equation involves domain integrals. To avoid domain integrals, the dual reciprocity method (DRM) was proposed [5]. This method actually divides the solution into two parts: a particular solution of the inhomogeneous PDE plus a solution of its homogeneous counterpart. Since particular solutions

to complex inhomogeneities are very difficult or even impossible to obtain, the inhomogeneity is normally represented by a series expansion in terms of simpler functions for which particular solutions can be (easily) determined [11]. Furthermore, the DRM has been extended to deal with nonlinear problems, heterogeneous problems, variable coefficient problems and time-dependent problems.

In this paper, we first consider the following equation

$$\nabla^2 u = p(u), \quad (1)$$

where  $p$  is a linear operator. This problem has been addressed in [5], but we will present a new formulation which turns out to be easier and more efficient. We have to say something about the type of equation (1). Since  $p$  is a linear operator with respect to  $u$ , one might hope to be able to find the fundamental solution to the (homogeneous) equation (1); however, this is normally difficult, even more when  $p(u)$  includes variable coefficients. To solve equation (1), we may employ the fundamental solution for the Laplacian, and consider the problem as a Poisson-type equation. Therefore the right hand side  $p(u)$  may be regarded as an inhomogeneity or *pseudo source*.

Secondly we apply the dual reciprocity method to the following axisymmetric Poisson equation.

$$\nabla_a^2 u := u_{rr} + u_r/r + u_{zz} = b, \quad (2)$$

where  $b$  is a given function. Conceptually this is nothing special, but it is extremely difficult to find a particular solution to the axisymmetric Laplacian for the traditional radial basis functions (RBFs) being the right hand side. We turned to the axisymmetric RBFs.

The paper is organised as follows: In Section 2, we recall the essentials of the DRM and show how equation (1) is solved by the DRM traditionally. A new formulation for equation (1) is derived in Section 3, which is better than the traditional one. In Section 4, an axisymmetric DRM formulation is established for equation (2). A few numerical examples are shown in Section 5. Comments and conclusions are given in Section 6.

## 2 The basic DRM

In this section we briefly recall the essentials of the DRM. We start with the Laplace equation. Consider a domain  $\Omega$  with boundary  $\Gamma$  and let  $u$  satisfy

$$\nabla^2 u = 0. \quad (3)$$

Then the integral equation corresponding to (3) is

$$c(\mathbf{x})u(\mathbf{x}) + \int_{\Gamma} q^* u d\Gamma = \int_{\Gamma} q u^* d\Gamma, \quad (4)$$

where  $c(\mathbf{x})$  is a function of position;  $u^*$  and  $q^*$  are the fundamental solution and its normal derivative;  $q$  is the normal derivative of  $u$ .

We subdivide the boundary  $\Gamma$  into boundary elements by  $N$  nodes  $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ , and if we have  $L$  points  $\{\mathbf{x}_{N+1}, \dots, \mathbf{x}_{N+L}\}$  inside  $\Omega$  for which the values of  $u$  should be evaluated, then the BEM discretised version for (4) reads

$$\mathbf{H}\mathbf{u} - \mathbf{G}\mathbf{q} = 0, \quad (5)$$

where  $\mathbf{u} := (u(\mathbf{x}_1), \dots, u(\mathbf{x}_{N+L}))^T$ ,  $\mathbf{q} := (q(\mathbf{x}_1), \dots, q(\mathbf{x}_N))^T$ . The matrices  $\mathbf{H}$  and  $\mathbf{G}$  have the following structure

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{0} \\ \mathbf{H}_2 & \mathbf{I} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}, \quad (6)$$

where  $\mathbf{H}_1$  and  $\mathbf{G}_1$  are  $N \times N$  matrices,  $\mathbf{H}_2$  and  $\mathbf{G}_2$  are  $L \times N$  matrices,  $\mathbf{I}$  is an  $L \times L$  unit matrix.

we deliberately put two uncoupled equations in the form (5) to make the writing simpler when we talk about the DRM later.

## 2.1 DRM for Poisson equations

Consider a simple source term for the Poisson equation

$$\nabla^2 u = b, \quad (7)$$

where  $b$  is some smooth function of  $\mathbf{x}$ .

To solve this equation,  $b$  is first approximated in terms of a set of (radial) basis functions  $\{\phi_j\}$

$$b(\mathbf{x}) \doteq \sum_j^{N+L} \alpha_j \phi_j(\mathbf{x}). \quad (8)$$

To determine the coefficients  $\alpha_j$ , we use interpolation, i.e.

$$b(\mathbf{x}_i) = \sum_j^{N+L} \alpha_j \phi_j(\mathbf{x}_i), \quad i = 1, \dots, N+L. \quad (9)$$

In matrix form, this reads

$$\mathbf{F}\boldsymbol{\alpha} = \mathbf{b}, \quad (10)$$

where  $\mathbf{b} := (b(\mathbf{x}_1), \dots, b(\mathbf{x}_{N+L}))^T$ ,  $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_{N+L})^T$  and the matrix  $\mathbf{F}$  is defined by its elements  $\phi_j(\mathbf{x}_i)$ .

We are interested in such  $\phi_j$  that it is easy to find a particular solution  $\hat{u}_j$  for which

$$\nabla^2 \hat{u}_j = \phi_j, \quad j = 1, \dots, N + L. \quad (11)$$

This implies that

$$\hat{u}(\mathbf{x}) = \sum_j^{N+L} \alpha_j \hat{u}_j(\mathbf{x}), \quad (12)$$

is an approximate particular solution to equation (7), i.e.,  $u - \hat{u}$  approximately satisfies the Laplace equation

$$\nabla^2(u - \hat{u}) = 0. \quad (13)$$

From the standard BEM (see (5)) we have

$$\mathbf{H}(\mathbf{u} - \hat{\mathbf{u}}) - \mathbf{G}(\mathbf{q} - \hat{\mathbf{q}}) = 0, \quad (14)$$

i.e.

$$\mathbf{H}\mathbf{u} - \mathbf{G}\mathbf{q} = \mathbf{H}\hat{\mathbf{u}} - \mathbf{G}\hat{\mathbf{q}}, \quad (15)$$

where  $\hat{\mathbf{u}}$  and  $\hat{\mathbf{q}}$  are defined similarly to  $\mathbf{u}$  and  $\mathbf{q}$  respectively.

From (12), we can easily see that

$$\hat{\mathbf{u}} = \hat{\mathbf{U}}\boldsymbol{\alpha}, \quad \hat{\mathbf{q}} = \hat{\mathbf{Q}}\boldsymbol{\alpha}, \quad (16)$$

where the matrices  $\hat{\mathbf{U}}$  and  $\hat{\mathbf{Q}}$  are  $(\hat{u}_j(\mathbf{x}_i))$  and  $(\hat{q}_j(\mathbf{x}_i))$  respectively.

By plugging (16) into (15) and using (10) we finally arrive at

$$\mathbf{H}\mathbf{u} - \mathbf{G}\mathbf{q} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\mathbf{F}^{-1}\mathbf{b}. \quad (17)$$

This is the pith and marrow of the DRM.

## 2.2 DRM for pseudo-Possion equations

It is possible to generalise the foregoing to right hand sides which depend on the unknown

$$\nabla^2 u = p(u), \quad (18)$$

where  $p$  is a linear operator. Before we give our alternative approach in section 3, we first show how this kind of problem has been solved in [5].

Define the vector  $\mathbf{p} := [p(u)(\mathbf{x}_1), \dots, p(u)(\mathbf{x}_{N+L})]^T$ . Using equation (17), we obtain

$$\mathbf{H}\mathbf{u} - \mathbf{G}\mathbf{q} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\mathbf{F}^{-1}\mathbf{p}. \quad (19)$$

Now, in order to solve equation (19), the unknown  $\mathbf{p}$  should be represented by  $\mathbf{u}$  (and/or  $\mathbf{q}$ ). To this end,  $u$  is expanded in terms of  $\{\phi_j\}$

$$u(\mathbf{x}) \doteq \sum_j^{N+L} \alpha_j \phi_j(\mathbf{x}). \quad (20)$$

Subsequently we get

$$p(u)(\mathbf{x}) \doteq \sum_j^{N+L} \alpha_j p(\phi_j)(\mathbf{x}). \quad (21)$$

By using interpolation in (20) and (21), we obtain

$$u(\mathbf{x}_i) = \sum_j^{N+L} \alpha_j \phi_j(\mathbf{x}_i), p(u)(\mathbf{x}_i) = \sum_j^{N+L} \alpha_j p(\phi_j)(\mathbf{x}_i), i = 1, \dots, N + L. \quad (22)$$

Rewriting (22) in matrix form gives

$$\mathbf{u} = \mathbf{F}\boldsymbol{\alpha}, \quad \mathbf{p} = \mathbf{F}_1\boldsymbol{\alpha}, \quad (23)$$

where  $\mathbf{F}_1$  denotes the matrix  $(p(\phi_j)(\mathbf{x}_i))$ . From (23) we find that  $\mathbf{p}$  is given by

$$\mathbf{p} = \mathbf{F}_1\mathbf{F}^{-1}\mathbf{u}. \quad (24)$$

We now substitute this representation into (19) and obtain the system

$$\mathbf{H}\mathbf{u} - \mathbf{G}\mathbf{q} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\mathbf{F}^{-1}\mathbf{F}_1\mathbf{F}^{-1}\mathbf{u}, \quad (25)$$

from which  $\mathbf{u}$  and  $\mathbf{q}$  can be found after applying the boundary condition(s).

### 3 An easier DRM formulation

The main idea of the DRM is to expand the right hand side to find an approximate particular solution.

If we could find particular solutions  $\bar{u}_j$ , such that

$$\nabla^2 \bar{u}_j(\mathbf{x}) = p(\phi_j)(\mathbf{x}), \quad j = 1, \dots, N + L, \quad (26)$$

then we see from (20) and (21) that

$$\bar{u}(\mathbf{x}) := \sum_j^{N+L} \alpha_j \bar{u}_j(\mathbf{x}), \quad (27)$$

satisfies

$$\nabla^2 \bar{u} \doteq p(u), \quad (28)$$

i.e.,  $u - \bar{u}$  satisfies Laplace's equation. Consequently, by using (5) we obtain

$$\mathbf{H}\mathbf{u} - \mathbf{G}\mathbf{q} = \mathbf{H}\bar{\mathbf{u}} - \mathbf{G}\bar{\mathbf{q}}. \quad (29)$$

Define matrices  $\bar{\mathbf{U}}_{ij} = (\bar{u}_j(\mathbf{x}_i))$  and  $\bar{\mathbf{Q}}_{ij} = (\bar{q}_j(\mathbf{x}_i))$ . Performing the same manipulations as in Section 2.1, we get a more elegant computational scheme

$$\mathbf{H}\mathbf{u} - \mathbf{Q}\mathbf{q} = (\mathbf{H}\bar{\mathbf{U}} - \mathbf{G}\bar{\mathbf{Q}})\mathbf{F}^{-1}\mathbf{u}. \quad (30)$$

Comparing (30) with (25), we can see (30) is easier and more efficient than (25). Scheme (30) looks more like (17), the difference is that  $\hat{\mathbf{U}}$  and  $\hat{\mathbf{Q}}$  are replaced by  $\bar{\mathbf{U}}$  and  $\bar{\mathbf{Q}}$  respectively. While scheme (25) looks a bit more complicated.

**Remark 1.** We can see that, the essential difference between this new approach and the traditional DRM is that the new approach uses the particular solutions  $\bar{u}_j$

$$\nabla^2 \bar{u}_j = p(\phi_j), \quad (31)$$

while the traditional DRM uses the particular solutions  $\hat{u}_j$

$$\nabla^2 \hat{u}_j = \phi_j. \quad (32)$$

For radial basis functions (RBFs) [7],  $\hat{u}_j$  are always available for the Laplacian, but the availability of  $\bar{u}_j$  depends on the operator  $p$ . If it is really difficult to find  $\bar{u}_j$ , then we can choose  $\{\phi_j\}$  such that  $p(\phi_j)$  are RBFs. In this way,  $\bar{u}_j$  is also nearly always available.

**Remark 2.** For the case  $p(u) = cu$  ( $c$  is a constant), it is obvious that scheme (30) and (25) result in the same set of linear equations since  $\mathbf{F}_1 = c\mathbf{F}$  and  $\hat{u}_j = c\bar{u}_j$ .

## 4 An axisymmetric DRM formulation

The major problem here is to find a particular solution  $\bar{u}_j$  for a given basis function  $\phi_j$  (e.g. a radial basis function), satisfying

$$\nabla_a^2 \bar{u}_j = \phi_j. \quad (33)$$

This is not a trivial task. To circumvent this problem, one may start with a given  $\bar{u}_j$  and then compute  $\phi_j$  by differentiation. For example, in [10] the particular solution is chosen to be  $\bar{u}_j = d_j^3/12$ ; therefore the basis function is then  $\phi_j = d_j(1 - r_j/4r)$ . In [4],  $\bar{u}_j = r_j(Cd_j^3 + r^3d_j^2)$  and  $\phi_j = r_j(C(12 - 3r_j/r)d_j - 14r_jr^2 + 9rd_j^2 + 18r^3)$ . We can see that the choice of  $\bar{u}_j$  is rather arbitrary and lacks mathematical foundation. Another weak point of these two methods is that  $\phi_j$  is not defined at  $r = 0$ .

Here we develop a simple and logical method which enables us to find the analytical expression for both  $\bar{u}_j$  and  $\phi_j$  satisfying (33). The idea is to use a three-dimensional particular solution and integrate it with respect to  $\theta$  from 0 to  $2\pi$ . This method has been explored in [6] for the Helmholtz equation. In that case,  $\bar{u}_j$  is expressed by an integral. Fortunately, for the Laplacian we are able to derive  $\bar{u}_j$  in a closed form, at least for the linear RBF.

In the three-dimensional case we know that  $\bar{u}'_j = \frac{d_j^3}{12}$  satisfies

$$\frac{\partial^2 \bar{u}'_j}{\partial x^2} + \frac{\partial^2 \bar{u}'_j}{\partial y^2} + \frac{\partial^2 \bar{u}'_j}{\partial z^2} = d_j. \quad (34)$$

Rewriting this equation in the cylindrical coordinates  $(r, \theta, z)$  gives

$$\frac{1}{r^2} \frac{\partial^2 \bar{u}'_j}{\partial \theta^2} + \frac{\partial^2 \bar{u}'_j}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}'_j}{\partial r} + \frac{\partial^2 \bar{u}'_j}{\partial z^2} = d_j. \quad (35)$$

Integrating the above equation, we obtain

$$\frac{\partial^2 \bar{u}_j}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_j}{\partial r} + \frac{\partial^2 \bar{u}_j}{\partial z^2} = \phi_j, \quad (36)$$

where

$$\phi_j = \int_0^{2\pi} d_j d\theta = 4\sqrt{a+b}E(k), \quad (37)$$

and

$$\bar{u}_j = \int_0^{2\pi} \frac{d_j^3}{12} d\theta = \frac{1}{9}(a+b)\sqrt{a+b} \left( (k^2-1)K(k) + (4-2k^2)E(k) \right). \quad (38)$$

Here  $a := r_j^2 + r^2 + (z_j - z)^2$ ,  $b := 2r_j r$ ,  $k := \sqrt{2b/(a+b)}$ ;  $K$  and  $E$  are the complete elliptic function of the first kind and second kind, respectively. The first order derivatives of  $\bar{u}_j$  are

$$\frac{\partial \bar{u}_j}{\partial r} = (r+r_j)\sqrt{a+b}E(k) + \frac{2r_j}{3k^2}\sqrt{a+b} \left( (k^2-1)K(k) + (1-2k^2)E(k) \right), \quad (39)$$

and

$$\frac{\partial \bar{u}_j}{\partial z} = (z-z_j)\sqrt{a+b}E(k). \quad (40)$$

It is easy to verify that  $\phi_j$ ,  $\bar{u}_j$ ,  $\frac{\partial \bar{u}_j}{\partial r}$  and  $\frac{\partial \bar{u}_j}{\partial z}$  are all continuous in the whole domain  $\{(r, z) \mid r \geq 0, -\infty < z < \infty\}$ .

In [6], the author has numerically shown the local property of  $\phi_j$ . This is important when the function to be interpolated varies steeply over the domain.

## 5 Numerical examples

In this section, we demonstrate some numerical examples.

### 5.1 A convection–diffusion problem

As a first application, let us investigate a steady 2D convection–diffusion problem in Cartesian coordinates

$$\mathcal{K}\nabla^2 u = v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y}, \quad (41)$$

where  $u$  is the quantity being convected,  $v_1$  and  $v_2$  are the velocity components, and  $\mathcal{K}$  is the diffusivity. Combining  $v_1$ ,  $v_2$  and  $\mathcal{K}$  into parameters  $a$  and  $b$  respectively, we obtain

$$\nabla^2 u = a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y}. \quad (42)$$

We assume here that  $a$  and  $b$  are both constant.

In this case,  $p = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$ . Our task is to find  $\bar{u}_j$  such that

$$\nabla^2 \bar{u}_j = a \frac{\partial \phi_j}{\partial x} + b \frac{\partial \phi_j}{\partial y}, \quad (43)$$

for each  $\phi_j$ .

It is easy to *verify* (although maybe less easy to *find*) that, for the linear radial basis function

$$\phi_j := d_j, \quad (44)$$

with  $d_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$ , the particular solution is given by

$$\bar{u}_j = (ax + by)d_j/3. \quad (45)$$

For the *thin plate spline*

$$\phi_j := d_j^2 \ln(d_j), \quad (46)$$

the particular solution is

$$\bar{u}_j = (ax + by)d_j^2(4 \ln(d_j) - 1)/16. \quad (47)$$

For variable  $a$  and  $b$ , we can't expect that we can always find the particular solutions  $\bar{u}_j$  for commonly used RBFs. If this is the case, then **remark 1** in the previous section applies.

we consider a model problem in an oval domain ( $x^2/4 + y^2 < 1$ )

$$\nabla^2 u = -\frac{\partial u}{\partial x} \quad (48)$$

i.e.  $a = -1$ ,  $b = 0$ . the boundary condition is chosen such that the problem has an exact solution  $u = \exp(-x)$ .

The boundary is subdivided into elements by 16 boundary points; moreover we also use 17 internal points as in [5] (see Figure 1). The values of  $u$  at numbered points are shown in Table 1. The advantages of the new approach are obvious.

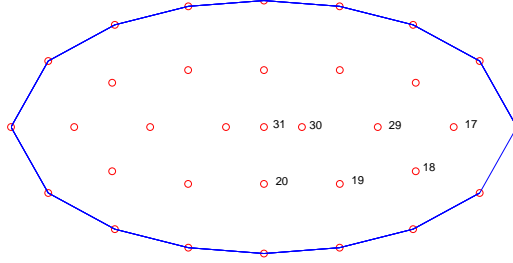


Figure 1: An Example  $\nabla^2 u = -\frac{\partial u}{\partial x}$

## 5.2 An axisymmetric Poisson equation

On a cylinder  $\Omega := \{(r, z) \mid 0 < r < 1, 0 < z < 1\}$ , consider the following test problem

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + u_{zz} = -\frac{\sin(r)}{r} \exp(-z), & 0 < r < 1, 0 < z < 1, \\ u = \cos(r), & z = 0, \\ u = \frac{\cos(r)}{e}, & z = 1, \\ q = -\sin(1) \exp(-z), & r = 1. \end{cases} \quad (49)$$

Actually, the boundary conditions are chosen in such a way that the problem has the exact solution  $u = \cos(r) \exp(-z)$ . We invest 15 linear elements on the boundary and 20 internal points for interpolation (including 4 points on  $z$ -axis). Results are shown in Table 2. Employing the basis functions and particular solutions derived in this section, we are able to put some interpolation points on the  $z$ -axis, this is computationally convenient and will improve accuracy. As we can see the numerical result agrees with the exact solution quite well.

## 5.3 An axisymmetric heat transfer problem

Obviously, the axisymmetric DRM formulation in Section 4 can be extended to the axisymmetric Poisson equation with unknown right hand side and hence can be used to solve the axisymmetric heat equation. we would like to demonstrate this by an example. Consider a cylinder  $\{(r, z) \mid 0 \leq r < 1, 0 < z < 1\}$  with initial temperature  $u_0$ , which is placed in a configuration for which the lower medium has temperature  $u_{1\infty}$  and the upper medium has temperature  $u_{2\infty}$  as shown in Fig-

node	x	y	exact	result[5]	error [5]	result	error
u17	1.5	0.0	0.223	0.229	0.006	0.222	0.001
u18	1.2	-0.35	0.301	0.307	0.006	0.301	<0.001
u19	0.6	-0.45	0.549	0.555	0.006	0.549	<0.001
u20	0.0	-0.45	1.000	1.003	0.003	1.001	0.001
u21	-0.6	-0.45	1.822	1.819	0.003	1.824	0.002
u22	-1.2	-0.35	3.320	3.323	0.003	3.320	<0.001
u23	-1.5	0.0	4.482	4.489	0.007	4.468	0.014
u29	0.9	0.0	0.406	0.441	0.035	0.407	0.001
u30	0.3	0.0	0.741	0.745	0.004	0.742	0.001
u31	0.0	0.0	1.000	1.002	0.002	1.001	0.001
u32	-0.3	0.0	1.350	1.348	0.002	1.351	0.001
u33	-0.9	0.0	2.460	2.448	0.012	2.461	0.001

Table 1: Results of  $\nabla^2 u = -\frac{\partial u}{\partial x}$

ure 2. A Robin boundary condition is specified for this example by

$$q = h(u_\infty - u), \quad (50)$$

where  $h$  is the contact conductance. We impose a Robin boundary condition because other types of boundary conditions are included as a special case. For instance, a Dirichlet boundary condition is a special case of (50) for large  $h$ , while a Neumann boundary condition can be considered as (50) with fairly small  $h$ . Moreover, sometimes, it is difficult to specify a Dirichlet boundary condition near interfaces due to temperature jumps, while a Robin boundary condition can overcome such difficulties elegantly.

Hence we have the following initial boundary value problem

$$\begin{cases} \nabla_a^2 u = \frac{1}{k} \frac{\partial u}{\partial t}, & 0 < r < 1, 0 < z < 1 \text{ and } t > 0, \\ q = h_1(u_{1\infty} - u), & z = 0 \text{ or } r = 1, \\ q = h_2(u_{2\infty} - u), & z = 1 \\ u = u_0, & t = 0. \end{cases} \quad (51)$$

We have tested a uniform grid with  $N = 30$  and  $L = 90$  and a time step size

node	r	z	exact solution	numerical solution	error
$u_{17}$	0	0.2	0.818731	0.817896	0.000834588
$u_{18}$	0	0.4	0.67032	0.669699	0.000620611
$u_{19}$	0	0.6	0.548812	0.548351	0.000460235
$u_{20}$	0	0.8	0.449329	0.448992	0.000336916
$u_{21}$	0.2	0.2	0.802411	0.801613	0.000797736
$u_{22}$	0.2	0.4	0.656958	0.656351	0.000606854
$u_{23}$	0.2	0.6	0.537872	0.537419	0.000452663
$u_{24}$	0.2	0.8	0.440372	0.440046	0.000326601
$u_{25}$	0.4	0.2	0.754101	0.753366	0.000735144
$u_{26}$	0.4	0.4	0.617406	0.616834	0.000571212
$u_{27}$	0.4	0.6	0.505489	0.505054	0.000435442
$u_{28}$	0.4	0.8	0.413859	0.413548	0.000311157
$u_{29}$	0.6	0.2	0.675728	0.6751	0.000627978
$u_{30}$	0.6	0.4	0.553239	0.55272	0.000519052
$u_{31}$	0.6	0.6	0.452954	0.452533	0.000420779
$u_{32}$	0.6	0.8	0.370847	0.370551	0.000295912
$u_{33}$	0.8	0.2	0.570415	0.569985	0.000430029
$u_{34}$	0.8	0.4	0.467016	0.466533	0.000483745
$u_{35}$	0.8	0.6	0.382361	0.381909	0.000451466
$u_{36}$	0.8	0.8	0.313051	0.312748	0.000302453

Table 2: Results of  $\nabla_a^2 u = -\frac{\sin(r)}{r} \exp(-z)$ .

$\Delta t = 0.01$ . The parameters in the example are chosen to be

$$\begin{cases} k = 1, \\ h_1 = h_2 = 1, \\ u_{1\infty} = 0, \\ u_{2\infty} = 0.5, \\ u_0 = 1. \end{cases} \quad (52)$$

The temperature distributions, as the time marches forward, are shown in Figures 3(a) to 3(g). Note that the temperature on the interface is neither  $u_{1\infty}$  or  $u_{2\infty}$  but something in between.

The equilibrium temperature as  $t \rightarrow \infty$  can be found by solving the following

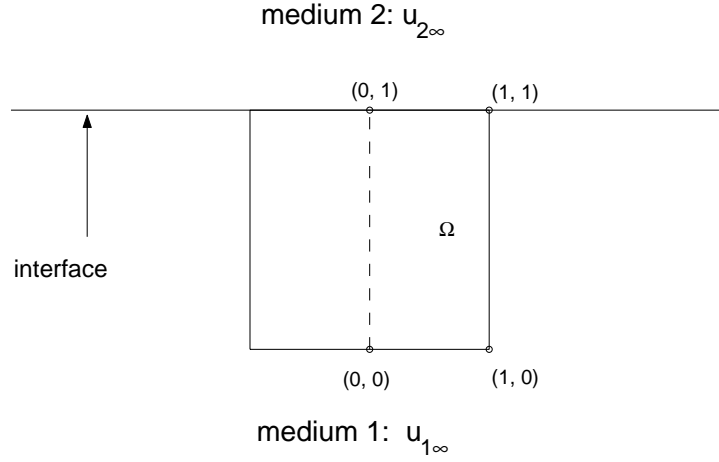


Figure 2: The heat transfer problem.

static problem

$$\begin{cases} \nabla_a^2 u = 0, & 0 < r < 1, 0 < z < 1, \\ q = h_1(u_{1\infty} - u), & z = 0 \text{ or } r = 1, \\ q = h_2(u_{2\infty} - u), & z = 1. \end{cases} \quad (53)$$

The result is shown in Figure 3 (h), where we can see that it agrees very nicely with the temperature when  $t = 2.5$ .

## 6 Conclusions

Two DRM formulations are derived and their efficiency and accuracy are demonstrated by a number of numerical examples.

For a pseudo-Poisson equation with the right hand side being a linear operation on the unknown, the new DRM formulation is easier and more efficient.

In the axisymmetric DRM formulation, the particular solution is written in a closed form by using the linear axisymmetric radial basis function. Obviously this formulation can be extended to axisymmetric nonlinear problems and axisymmetric time-dependent problems.

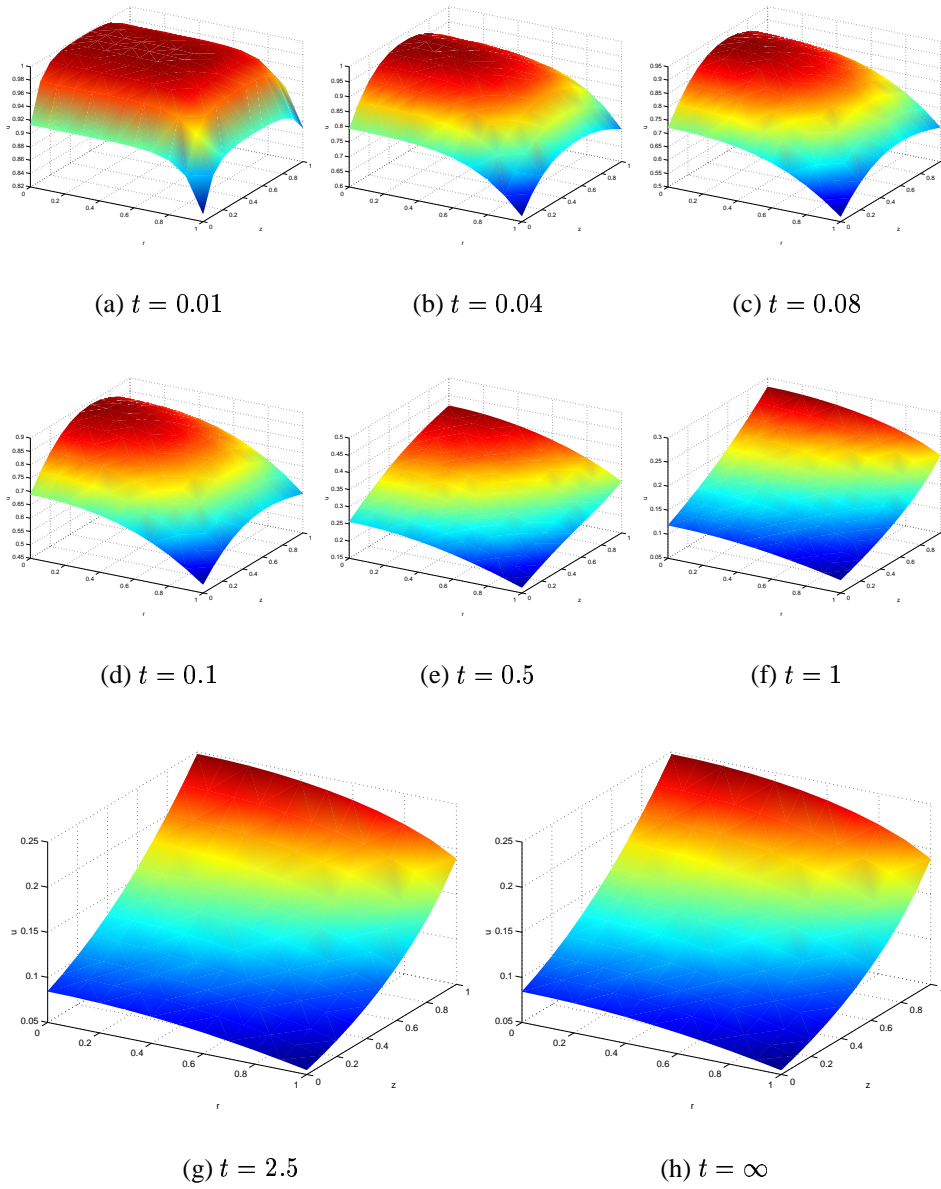


Figure 3: Results of the heat transfer problem.

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