

TREATMENT OF CORNERS IN THE DUAL RECIPROCIITY BOUNDARY ELEMENT METHOD(DRM)

R.M.M. MATTHEIJ, H.G.TER MORSCHE, P.J.P.M SIMONS, AND K. WANG

ABSTRACT. This paper, from the computational point of view, discusses the treatment of corners in the context of the DRM. We first briefly recall discontinuous element method, then investigate two alternatives - Analytical Method and Galerkin DRM. Numerical results show they are efficient and viable in practice.

1. THE COMPUTATIONAL STRATEGY OF THE DRM

The key idea of the DRM[1] is expanding the inhomogeneity in terms of simpler functions for which particular solutions can be easily determined. Among various 'simpler functions' which have been proposed to approximate the inhomogeneity, the most popular ones are Radial Basis Functions (RBFs)[2]. For simplicity, let us consider the Poisson equation

$$(1.1) \quad \Delta u = f \quad \text{in } \Omega \subset R^2$$

together with a boundary condition on $\Gamma = \partial\Omega$, which could be of Dirichlet or Neumann type, also a combination of these types is possible.

Suppose

$$\hat{f}(x) = \sum_{j=1}^{N+L} \alpha_j \phi(\|x - x_j\|)$$

is the RBFs' approximation to $f(x)$, where $x_1, \dots, x_N \in \Gamma$ (boundary points), $x_{N+1}, \dots, x_{N+L} \in \Omega$ (internal points); and $\hat{u}_j(x)$ are the particular solutions of Poisson equations with source term $\phi(\|x - x_j\|)$, i.e., $\Delta \hat{u}_j(x) = \phi(\|x - x_j\|)$. Then, by Green's Identity, we have[1]

$$(1.2) \quad \begin{aligned} c(x)u(x) + \int_{\Gamma_y} \frac{u^*(x,y)}{\partial n_y} u(y) d\Gamma_y - \int_{\Gamma_y} u^*(x,y) \frac{\partial u(y)}{\partial n_y} d\Gamma_y \\ = c(x)\hat{u}(x) + \int_{\Gamma_y} \frac{u^*(x,y)}{\partial n_y} \hat{u}(y) d\Gamma_y - \int_{\Gamma_y} u^*(x,y) \frac{\partial \hat{u}(y)}{\partial n_y} d\Gamma_y, \end{aligned}$$

where $u^*(x,y) = -\frac{1}{2\pi} \log(\|x - y\|)$, $\hat{u}(x) = \sum_{j=1}^{N+L} \alpha_j \hat{u}_j(x)$, and

$$c(x) = \begin{cases} 1 & x \in \Omega \\ \frac{\theta}{2\pi} & x \in \Gamma \\ 0 & \text{otherwise} \end{cases}, \quad \text{here } \theta \text{ is the internal angle at boundary point } x \text{ in radians.}$$

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Now, the computational strategy of the DRM is, first, restricting x to the boundary Γ in (1.2), obtaining an BIE, solving the BIE, getting boundary data, then letting x be in Ω , using (1.2) and the boundary data to compute $u(x)$ (sometimes, and also the derivatives of $u(x)$) in Ω . In the remainder of this paper, we simply call the former step computing process and the latter step post process.

If the flux $\frac{\partial u(x)}{\partial n}$ is continuous on Γ , then the IE (1.2) can be formulated into the matrix form by boundary element discretization[1]

$$(1.3) \quad \begin{bmatrix} H_1 & 0 \\ H_2 & I \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} Q = \left\{ \begin{bmatrix} H_1 & 0 \\ H_2 & I \end{bmatrix} \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \hat{Q} \right\} \alpha.$$

Here, the vectors and matrices in equation (1.3) are defined as

$$\begin{aligned} U_1, & \quad N \times 1, & U_1(i) &= u(x_i), x_i \in \Gamma; \\ U_2, & \quad L \times 1, & U_2(i) &= u(x_i), x_i \in \Omega; \\ Q, & \quad N \times 1, & Q(i) &= \frac{\partial u(x_i)}{\partial n}, x_i \in \Gamma; \\ \alpha, & \quad (N+L) \times 1, & \alpha(i) & \text{interpolation coefficients of RBFs;} \\ \hat{U}_1, & \quad N \times (N+L), & \hat{U}_1(i, j) &= \hat{u}_j(x_i), x_i \in \Gamma; \\ \hat{U}_2, & \quad L \times (N+L), & \hat{U}_2(i, j) &= \hat{u}_j(x_i), x_i \in \Omega; \\ \hat{Q}, & \quad N \times (N+L), & \hat{Q}(i, j) &= \frac{\partial \hat{u}_j(x_i)}{\partial n}, x_i \in \Gamma; \\ H_1, & \quad N \times N, & H_1(i, j) &= c(x_i) \delta_{ij} + \\ & & & \sum_{E(k,l)=j} \int_{\Gamma_k} \frac{\partial u^*(x_i, y)}{\partial n_y} \phi_l(y) d\Gamma_y, x_i \in \Gamma; \\ H_2, & \quad L \times N, & H_2(i, j) &= \sum_{E(k,l)=j} \int_{\Gamma_k} \frac{\partial u^*(x_i, y)}{\partial n_y} \phi_l(y) d\Gamma_y, x_i \in \Omega; \\ I, & \quad L \times L, & & \text{unit matrix of order } L \\ G_1, & \quad N \times N, & G_1(i, j) &= \sum_{E(k,l)=j} \int_{\Gamma_k} u^*(x_i, y) \phi_l(y) d\Gamma_y, x_i \in \Gamma; \\ G_2, & \quad L \times N, & G_2(i, j) &= \sum_{E(k,l)=j} \int_{\Gamma_k} u^*(x_i, y) \phi_l(y) d\Gamma_y, x_i \in \Omega; \end{aligned}$$

where N is the number of boundary points, L is the number of internal points, δ_{ij} is Kroneker notation, $E(k, l)$ maps the l -th local node of the k -th element to its global node. $\phi_l(y)$ is the basis function corresponding to the l -th local node; $c(x)$, $u_j(x)$, $u^*(x, y)$ are as mentioned previously.

The formula (1.3) is a very elegant one which includes both the computing process and the post process. Obviously, in the computing process we need the first equation

$$H_1 U_1 - G_1 Q = (H_1 \hat{U}_1 - G_1 \hat{Q}) \alpha;$$

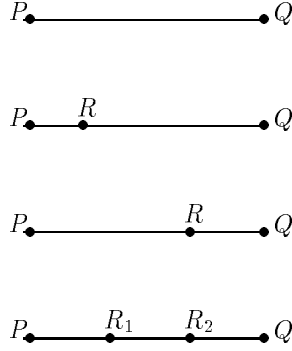


FIGURE 1. Types of Elements

and in the post process we use the second equation

$$U_2 + H_2 U_1 - G_2 Q = (\hat{U}_2 + H_2 \hat{U}_1 - G_2 \hat{Q}) \alpha.$$

But, if Γ has corners, then the flux $\frac{\partial u(x)}{\partial n}$ will be discontinuous at corners, therefore we can not use (1.3) directly. Some measures must be taken to handle this situation.

From now on, we assume the flux $\frac{\partial u(x)}{\partial n}$ has at most discontinuities of the first kind.

2. DISCONTINUOUS ELEMENTS

To treat corners, the discontinuous element method has been proposed[1][5]. Actually the ends of elements are not necessarily local nodes. In the discontinuous element method, if an element includes an end which is a corner, then we can avoid taking the end as an interpolation point just by ‘displacing’ the end into the element.

We take linear elements as examples, let PQ be a element, then we have four possibilities(see Figure 1)

- type (1):** P, Q are non-corner points, we take P and Q as local nodes.
- type (2):** only P is corner point, we first get a point R such that $PR/PQ = 1/K$ then take R and Q as local nodes.
- type (3):** only Q is corner point, we first get a point R such that $RQ/PQ = 1/K$ then take P and R as local nodes.
- type (4):** P, Q are both corners, we first get two points R_1 and R_2 such that $PR_1/PQ = 1/K$ and $R_2Q/PQ = 1/K$ then take R_1 and R_2 as local nodes.

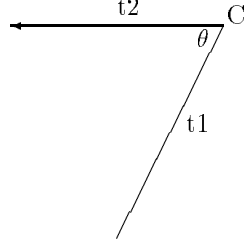


FIGURE 2. A Corner

The local basis functions are

$$\text{element of type (1): } \phi_1 = \frac{1-\xi}{2}, \quad \phi_2 = \frac{1+\xi}{2};$$

$$\text{element of type (2): } \phi_1 = \frac{K(1-\xi)}{2(K-1)}, \quad \phi_2 = \frac{K\xi+K-2}{2(K-1)};$$

$$\text{element of type (3): } \phi_1 = \frac{-K\xi+K-2}{2(K-1)}, \quad \phi_2 = \frac{K(1+\xi)}{2(K-1)};$$

$$\text{element of type (4): } \phi_1 = (1/2) - \frac{K\xi}{2(K-2)}, \quad \phi_2 = (1/2) + \frac{K\xi}{2(K-2)}.$$

Now, all nodes are non-corner points, then everything can be done according to steps of the standard DRM.

Remark1 The Element of type (4) can be avoided by partitioning the element into two elements.

Remark2 A suitable K is important to this method. Too large K and too small K both increase inaccuracy. A reasonable K is between 2 and 10.

3. ANALYTICAL METHOD

Actually, if a corner is not a cusp, i.e. $\theta \neq 0$, and the derivatives of the first order of u with respect to x and y are continuous on Γ (in many cases, the discontinuity of the flux at a corner is not because of the discontinuity of the derivatives, but of the fact that the unit outward normal has a jump at the corner), and if the original problem (1.1) is compatible, then we can determine analytically ‘the fluxes at the corner’, that is why we call this approach analytical method. We only consider Dirichlet boundary condition. Neumann problems are easy to handle because there is only one unknown per corner ([1] P_{26-27}).

Without loss of generality, let

$$x = x(s), \quad y = y(s), \quad 0 \leq s \leq S$$

be the parameterized representation of Γ , where s is the arc length relatively to a fixed point on Γ ; C be a corner on Γ with parameter $s = s_0$; t_1 and t_2 be two unit tangential directions at point C (pointing the positive direction of Γ) (see Figure 2). Then

$$(3.1) \quad \begin{aligned} \lim_{s \rightarrow s_0^-} \frac{dg(x(s), y(s))}{ds} &= \frac{\partial u(C)}{\partial t_1} = \left(\frac{\partial u(C)}{\partial x}, \frac{\partial u(C)}{\partial y} \right) \cdot t_1 \\ \lim_{s \rightarrow s_0^+} \frac{dg(x(s), y(s))}{ds} &= \frac{\partial u(C)}{\partial t_2} = \left(\frac{\partial u(C)}{\partial x}, \frac{\partial u(C)}{\partial y} \right) \cdot t_2 \end{aligned}$$

Now, (3.1) is a linear system with two unknowns $(\frac{\partial u(C)}{\partial x}, \frac{\partial u(C)}{\partial y})$. Because of $\theta \neq 0$, we can uniquely determine $(\frac{\partial u(C)}{\partial x}, \frac{\partial u(C)}{\partial y})$ from (3.1) since t_1, t_2 are linearly independent. Furthermore, the directional derivative of $u(x)$ along any direction t can be easily determined by $(\frac{\partial u(C)}{\partial x}, \frac{\partial u(C)}{\partial y}) \cdot t$, by means of this, both fluxes 'before' and 'after' the corner are determined.

Let NC be the number of corners on the boundary Γ , then we can get $2*NC$ equations according to the above procedure. That means we have two equations at every corner. These equations together with those at non-corners compose our desired linear system. What follows are the brief computational steps.

Computing process We want to compute the values of $q(x) := \partial u(x)/\partial n$ at boundary points. Suppose $Q_{(N+NC) \times 1}$ (a corner point has two degrees of freedom) is the vector we want to find, we put the values of $u(x)$ in the same way, we get vector $U_{1(N+NC) \times 1}$ (values repeat at corners). After discretizing the BIE, we still arrive at (each corner is regarded as two points)

$$(3.2) \quad H_1 U_1 - G_1 Q = (H_1 \hat{U}_1 - G_1 \hat{Q}) \alpha;$$

but, where

$$\begin{aligned} U_1, & \quad (N + NC) \times 1, & U_1(i) &= u(x_i), x_i \in \Gamma; \\ Q, & \quad (N + NC) \times 1, & Q(i) &= \frac{\partial u(x_i)}{\partial n}, x_i \in \Gamma; \\ \alpha, & \quad (N + L) \times 1, & \alpha(i) & \text{interpolation coefficients of RBFs}; \\ \hat{U}_1, & \quad (N + NC) \times (N + L), & \hat{U}_1(i, j) &= \hat{u}_j(x_i), x_i \in \Gamma; \\ \hat{Q}, & \quad (N + NC) \times (N + L), & \hat{Q}(i, j) &= \frac{\partial \hat{u}_j(x_i)}{\partial n}, x_i \in \Gamma; \\ H_1, & \quad (N + NC) \times (N + NC), & H_1(i, j) &= c(x_i) \delta_{ij} + \\ & & & \sum_{E(k,l)=j} \int_{\Gamma_k} \frac{\partial u^*(x_i, y)}{\partial n_y} \phi_l(y) d\Gamma_y, x_i \in \Gamma; \\ G_1, & \quad (N + NC) \times (N + NC), & G_1(i, j) &= \\ & & & \sum_{E(k,l)=j} \int_{\Gamma_k} u^*(x_i, y) \phi_l(y) d\Gamma_y, x_i \in \Gamma. \end{aligned}$$

In this set of equations, each boundary point corresponds to one equation (one corner point is regarded as two points). Now we change this set of equations according to the following procedure: if a boundary point is a non-corner point then keep the equation corresponding to it unchanged; if a boundary point P_i is a corner point then replace the equation corresponding to it with a new equation $Q(i) = (\frac{\partial u(P_i)}{\partial x}, \frac{\partial u(P_i)}{\partial y}) \cdot n$, here n is the unit outward normal at P_i (each corner has 'two n ').

Post process The matrix form of the post process is also

$$U_2 + H_2 U_1 - G_2 Q = (\hat{U}_2 + H_2 \hat{U}_1 - G_2 \hat{Q}) \alpha.$$

but, here

$$\begin{aligned}
U_1, & \quad (N + NC) \times 1, & U_1(i) &= u(x_i), x_i \in \Gamma; \\
U_2, & \quad L \times 1, & U_2(i) &= u(x_i), x_i \in \Omega; \\
Q, & \quad (N + NC) \times 1, & Q(i) &= \frac{\partial u(x_i)}{\partial n}, x_i \in \Gamma; \\
\alpha, & \quad (N + L) \times 1, & \alpha(i) & \text{interpolation coefficients of RBFs;} \\
\hat{U}_1, & \quad (N + NC) \times (N + L), & \hat{U}_1(i, j) &= \hat{u}_j(x_i), x_i \in \Gamma; \\
\hat{U}_2, & \quad L \times (N + L), & \hat{U}_2(i, j) &= \hat{u}_j(x_i), x_i \in \Omega; \\
\hat{Q}, & \quad (N + NC) \times (N + L), & \hat{Q}(i, j) &= \frac{\partial \hat{u}_j(x_i)}{\partial n}, x_i \in \Gamma; \\
H_2, & \quad L \times (N + NC), & H_2(i, j) &= \\
& & & \sum_{E(k,l)=j} \int_{\Gamma_k} \frac{\partial u^*(x_i, y)}{\partial n_y} \phi_l(y) d\Gamma_y, x_i \in \Omega; \\
I, & \quad L \times L, & & \text{unit matrix of order } L \\
G_2, & \quad L \times (N + NC), & G_2(i, j) &= \\
& & & \sum_{E(k,l)=j} \int_{\Gamma_k} u^*(x_i, y) \phi_l(y) d\Gamma_y, x_i \in \Omega;
\end{aligned}$$

Remark1 In practical computation, ‘the fluxes at a corner’ can be approximated by difference quotients.

Remark2 If linear elements are employed, the following formula is useful

$$\begin{aligned}
\int_{-1}^1 (a\xi + b) \log|\xi - c| d\xi &= (b + ac)(1 + c)(\log(1 + c) - 1) - \\
& (a/2)(1 + c)^2(\log(1 + c) - (1/2)) + \\
& (b + ac)(1 - c)(\log(1 - c) - 1) + \\
& (a/2)(1 - c)^2(\log(1 - c) - (1/2))
\end{aligned}$$

which can be used to compute singular integrations appearing in the DRM.

4. GALERKIN DRM

Another choice to solve the BIE is Galerkin method, we call this Galerkin DRM. It is easy to treat corners because more test functions get more equations. At each corner we have two test functions since the flux is discontinuous, but at the conventional boundary point we only have one test function.

The BIE is

$$\begin{aligned}
& c(x)u(x) + \int_{\Gamma_y} \frac{u^*(x, y)}{\partial n_y} u(y) d\Gamma_y - \int_{\Gamma_y} u^*(x, y) \frac{\partial u(y)}{\partial n_y} d\Gamma_y \\
& = c(x)\hat{u}(x) + \int_{\Gamma_y} \frac{u^*(x, y)}{\partial n_y} \hat{u}(y) d\Gamma_y - \int_{\Gamma_y} u^*(x, y) \frac{\partial \hat{u}(y)}{\partial n_y} d\Gamma_y, x \in \Gamma,
\end{aligned}$$

Suppose the test and admission space are both

$$span[\phi_1, \phi_2, \dots, \phi_{N+NC}],$$

and

$$u = \sum_{j=1}^{N+NC} U(j)\phi_j, \partial u/\partial n = \sum_{j=1}^{N+NC} Q(j)\phi_j.$$

Then Galerkin procedure yields

$$(4.1) \quad (H_1 + H_2)U - GQ = [(H_1 + H_2)\hat{U} - G\hat{Q}]\alpha$$

where

$$\begin{aligned} U, & \quad (N + NC) \times 1, U(i) = u(x_i), x_i \in \Gamma; \\ Q, & \quad (N + NC) \times 1, Q(i) = \frac{\partial u(x_i)}{\partial n}, x \in \Gamma; \\ \alpha, & \quad (N + L) \times 1, \alpha(i) \text{ interpolation coefficients of RBFs}; \\ \hat{U}, & \quad (N + NC) \times (N + L), \hat{U}(i, j) = \hat{u}_j(x_i), x_i \in \Gamma; \\ \hat{Q}, & \quad (N + NC) \times (N + L), \hat{Q}(i, j) = \frac{\partial \hat{u}_j(x_i)}{\partial n}, x_i \in \Gamma; \\ H_1, & \quad (N + NC) \times (N + NC), H_1(i, j) = \\ & \quad \sum_{\substack{E(k, l_1) = i \\ E(k, l_2) = j}} \int_{\Gamma_k} c(x)\phi_{l_1}(x)\phi_{l_2}(x)d\Gamma_x; \\ H_2, & \quad (N + NC) \times (N + NC), H_2(i, j) = \\ & \quad \sum_{\substack{E(k, l_1) = i \\ E(l, l_2) = j}} \int_{\Gamma_l} \int_{\Gamma_k} \frac{\partial u^*(x, y)}{\partial n_y} \phi_{l_1}(x)\phi_{l_2}(y)d\Gamma_x d\Gamma_y; \\ G, & \quad (N + NC) \times (N + NC), G(i, j) = \\ & \quad \sum_{\substack{E(k, l_1) = i \\ E(l, l_2) = j}} \int_{\Gamma_l} \int_{\Gamma_k} u^*(x, y)\phi_{l_1}(x)\phi_{l_2}(y)d\Gamma_x d\Gamma_y \end{aligned}$$

where the meanings of the notations are the same as mentioned previously.

Numerical integration To build H_2 and G , we must handle the integrals with the following forms

$$\begin{aligned} I_1 &= \int_{\Gamma_l} \int_{\Gamma_k} \frac{\partial u^*(x, y)}{\partial n_y} \phi_{l_1}(x)\phi_{l_2}(y)d\Gamma_x d\Gamma_y \\ I_2 &= \int_{\Gamma_l} \int_{\Gamma_k} u^*(x, y)\phi_{l_1}(x)\phi_{l_2}(y)d\Gamma_x d\Gamma_y \end{aligned}$$

As for elements of higher order, we refer readers to [3], we now briefly explain how to calculate I_1 and I_2 in the case of linear elements. Suppose the ends of Γ_k are w_1 and w_2 , and the ends of Γ_l are z_1 and z_2 , then by transformations

$$x = \frac{w_2 + w_1}{2} + \frac{w_2 - w_1}{2}\xi$$

and

$$y = \frac{z_2 + z_1}{2} + \frac{z_2 - z_1}{2}\eta$$

we transfer I_1 and I_2 to the standard forms

$$\begin{aligned} I_1 &= \frac{m(\Gamma_k)m(\Gamma_l)}{4} \int_{-1}^1 \int_{-1}^1 \frac{\partial u^*(x(\xi), y(\eta))}{\partial n_y} \phi_{l_1}(\xi) \phi_{l_2}(\eta) d\xi d\eta \\ I_2 &= \frac{m(\Gamma_k)m(\Gamma_l)}{4} \int_{-1}^1 \int_{-1}^1 u^*(x(\xi), y(\eta)) \phi_{l_1}(\xi) \phi_{l_2}(\eta) d\xi d\eta \end{aligned}$$

where, $\phi_{l_1}(\xi), \phi_{l_2}(\eta)$ are the basis functions on the standard element $[-1, 1]$, $m(\Gamma_k), m(\Gamma_l)$ are the length of Γ_k and Γ_l , respectively.

we have three situations

(1) if $\Gamma_k = \Gamma_l$, then

$$I_1 = 0$$

and I_2 can be also analytically calculated by means of the formula

$$\int_{-1}^1 \int_{-1}^1 \log|\xi - \eta|(a\xi + b)(c\eta + d) d\xi d\eta = (4\log 2 - 6)bd - ac.$$

(2) if $\Gamma_k \cap \Gamma_l = \phi$ then we employ Gaussian quadrature directly.

(3) if Γ_k and Γ_l are neighbours, this means the integrands in I_1 and I_2 have singularities at point $(\xi, \eta) = (1, -1)$ or $(\xi, \eta) = (-1, 1)$. To compute I_1 and I_2 , we first use Telles' transformation[1]

$$(\xi, \eta) = ((-1/2)t^2 + t + (1/2), (1/2)s^2 + s - (1/2))$$

or

$$(\xi, \eta) = ((1/2)t^2 + t - (1/2), (-1/2)s^2 + s + (1/2))$$

which force the Jacobian to be zero at the singularities, then Gaussian quadrature can be employed to compute

$$\begin{aligned} I_1 &= \frac{m(\Gamma_k)m(\Gamma_l)}{4} \times \\ &\int_{-1}^1 \int_{-1}^1 \frac{\partial u^*(x(\xi(t)), y(\eta(s)))}{\partial n_y} \phi_{l_1}(\xi(t)) \phi_{l_2}(\eta(s)) (1-t)(1+s) dt ds \\ I_2 &= \frac{m(\Gamma_k)m(\Gamma_l)}{4} \times \\ &\int_{-1}^1 \int_{-1}^1 u^*(x(\xi(t)), y(\eta(s))) \phi_{l_1}(\xi(t)) \phi_{l_2}(\eta(s)) (1-t)(1+s) dt ds \end{aligned}$$

or

$$\begin{aligned} I_1 &= \frac{m(\Gamma_k)m(\Gamma_l)}{4} \times \\ &\int_{-1}^1 \int_{-1}^1 \frac{\partial u^*(x(\xi(t)), y(\eta(s)))}{\partial n_y} \phi_{l_1}(\xi(t)) \phi_{l_2}(\eta(s)) (1+t)(1-s) dt ds \\ I_2 &= \frac{m(\Gamma_k)m(\Gamma_l)}{4} \times \\ &\int_{-1}^1 \int_{-1}^1 u^*(x(\xi(t)), y(\eta(s))) \phi_{l_1}(\xi(t)) \phi_{l_2}(\eta(s)) (1+t)(1-s) dt ds \end{aligned}$$

As for the post processing, it is completely the same as before.

5. NUMERICAL EXAMPLES

We take $\Omega = (0, 1) \times (0, 1)$, $f = e^x + 6y$, and

$$g = \begin{cases} 1 - (1/2)\log((p1 - x)^2 + p2^2) + e^x, & y = 0; \\ 2 - (1/2)\log((p1 - x)^2 + (p2 - 1)^2) + e^x, & y = 1; \\ 2 - (1/2)\log((p1^2 + (p2 - y)^2) + y^3, & x = 0; \\ 1 + e - (1/2)\log((p1 - 1)^2 + (p2 - y)^2) + y^3, & x = 1. \end{cases}$$

Here $(p1, p2)$ is a point outside the domain Ω .

It is trivial to verify this problem has exact solution

$$u = 1 - \log((p1 - x)^2 + (p2 - y)^2) + e^x + y^3.$$

Discretisation relative error $\|error\|_\infty$ is defined as

$$\|error\|_\infty = \max_{1 \leq i \leq n} \left| \frac{u_a(i) - u_e(i)}{u_e(i)} \right|$$

where u_a and u_e are approximate solution and exact solution respectively.

Table 1 shows the computational results. Columns, from leftmost to third column, are values of u inside the domain Ω computed by discontinuous element method, analytical method and Galerkin method, respectively. The rightmost column is exact solution. In computation, there are 5 linear elements on each side of the domain. To approximate f by RBFs, uniform grid is employed. We simply use linear RBF.

Table 2 gives the asymptotic behavior of methods. the leftmost column are partitioning parameters. If the number of elements on each side is m , we say the partitioning parameter is $m + 1$. Uniform grid is used to approximate f by RBFs.

dist.	ana.	Galerkin	exact
1.29636	1.29617	1.29633	1.29504
1.62209	1.62190	1.62194	1.62090
2.00689	2.00671	2.00676	2.00579
2.46287	2.46270	2.46280	2.46189
1.40775	1.40759	1.40796	1.40647
1.74030	1.74014	1.74021	1.73925
2.13276	2.13261	2.13265	2.13186
2.59721	2.59708	2.59729	2.59639
1.61404	1.61393	1.61428	1.61307
1.95407	1.95394	1.95380	1.95357
2.35542	2.35530	2.35513	2.35507
2.83017	2.83009	2.83029	2.82965
1.96191	1.96179	1.96160	1.96175
2.31016	2.31006	2.30925	2.31068
2.72162	2.72154	2.72074	2.72223
3.20848	3.20840	3.20815	3.20865

TABLE 1. Numerical Examples

partition	dist.	ana.	Galerkin
4	$3.56 \cdot 10^{-3}$	$2.79 \cdot 10^{-3}$	$3.05 \cdot 10^{-3}$
8	$4.20 \cdot 10^{-4}$	$3.80 \cdot 10^{-4}$	$5.66 \cdot 10^{-4}$
12	$1.50 \cdot 10^{-4}$	$1.47 \cdot 10^{-4}$	$2.65 \cdot 10^{-4}$
16	$8.44 \cdot 10^{-5}$	$8.36 \cdot 10^{-5}$	$1.62 \cdot 10^{-4}$

TABLE 2. Asymptotic behavior

6. CONCLUSION

To handle problems arising in the DRM because of corners, two methods - analytical method and Galerkin DRM - have been developed in the paper, the existent

method - discontinuous element method - is also covered to compare. Numerical examples show these methods are effective and powerful in practice.

Now we briefly give the features of these methods.

- None of these methods needs to compute $c(x)$ at corners. In discontinuous element method we ‘displace corners into elements’; in analytical method we don’t use equations at corners; in Galerkin DRM what we need is just $c(x) = (1/2), a.e.$
- Galerkin DRM is much slower than the others although it looks elegant. In Galerkin DRM we have to calculate double integrations rather than single integrations in order to build the set of linear equations, this is a heavy work, especially in cases of elements of higher order.
- All methods can be directly extended to three dimensional cases without essential difficulties;
- All techniques on choosing RBFs in the DRM are also applicable here.

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(R.M.M. Mattheij, H.G.ter Morsche, and K. Wang) MATHEMATICS DEPARTMENT, EINDHOVEN UNIVERSITY OF TECHNOLOGY, POSTBUS 513, 5600 MB EINDHOVEN, THE NETHERLANDS

(P.J.P.M Simons) TNO TPD GLASS TECHONLOGY, DEN DOLECH 2, POSTBUS 595, 5600 AN EINDHOVEN, THE NETHERLANDS

E-mail address, R.M.M. Mattheij: mattheij@win.tue.nl

E-mail address, H.G.ter Morsche: morscheh@win.tue.nl

E-mail address, P.J.P.M Simons: psimons@tpd.tno.nl

E-mail address, K. Wang: wang@win.tue.nl