Artin groups, Brauer algebras, and Tangle Combinatorics

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joint work with Dié Gijsbers & David Wales

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Contents

BMW algebra of type A_n

- BMW = Birman & Wenzl, Murakami
- BMW used for knot theory and classical quantum group reps
- Definition for type A_n carries over to any graph M

BMW algebra of type M over $\mathbb{Q}(l, m)$

$BMW(M)$:

generators

$$\{g_i \mid i \in M\}$$

relations

(braid1)	$g_i g_j = g_j g_i$	when	$i \not\sim j$,
(braid2)	$g_i g_j g_i = g_j g_i g_j$	when	$i \sim j$,
(skein)	$m e_i = l(g_i^2 + m g_i - 1)$	for all	i,
(self-intersection1)	$g_i e_i = l^{-1} e_i$	for all	i,
(self-intersection2)	$e_i g_j e_i = l e_i$	when	$i \sim j$.

Theorem*

$BMW(M)$ is finite-dimensional

$\Leftrightarrow$

connected components of M are in $\{\mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8 \mid n \in \mathbb{N}\}$.

$$\begin{aligned}
\dim BMW(\mathrm{A}_n) &= (n+1)!! \\
\dim BMW(\mathrm{D}_n) &= (2^n+1)n!! - (2^{n-1}+1)n! \\
\dim BMW(\mathrm{E}_6) &= 1,440,585 \\
\dim BMW(\mathrm{E}_7) &= 439,670,025 \\
\dim BMW(\mathrm{E}_8) &= 53,328,069,225
\end{aligned}$$

$n!!$ = the number of matchings of $2n$ points $= 1 \cdot 3 \cdots (2n-1)$

$*$ proviso: proof not yet written up

Links represented by diagrams

- ambient isotopy = {planar isotopy and Reidemeister moves}
- ambient isotopy does not change the represented link

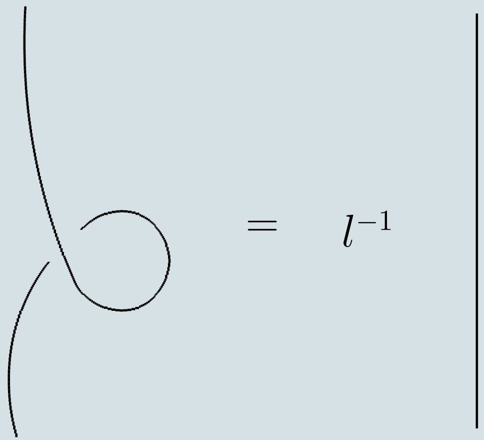
Theorem (Reidemeister) Two diagrams representing the same link are ambient isotopic.

Kauffman brackets

- $\mathbb{Q}[x, l^{\pm}, m]$-linear combinations of links
- modulo regular isotopy
- self-intersecting strands: factor l
- disjoint loops: factor x
- skein relation: parameter m

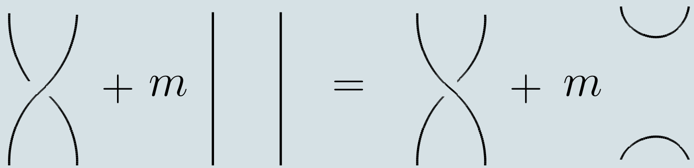
regular isotopy = {planar isotopy and Reidemeister moves II, III}

The self-intersection



Straightening at a cost of l^{-1}.
Other orientation at a cost of l.

The skein relation



Examples of Kauffman bracket computation

- two linked loops
- trefoil

Requirement

$$m(1-x) = l - l^{-1}$$

Theorem (Kauffman)

The above relations define a link invariant in $R := \mathbb{Q}(x)[l^{\pm}]$, after correction for writhe.

writhe = number of oriented crossings

Kauffman tangle algebra

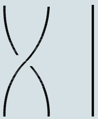
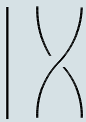
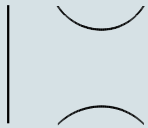
- An (n, k)-tangle is a rectangular slice of a diagram with n strands entering at the top and k strands ending at the bottom.
- Composition of tangles turns the set KaT_n of (n, n)-tangles into a ring
- An algebra over $KaT_0 = R$
- Closure of a tangle in KaT_n is a Kauffman bracket

$R = \mathbb{Q}(x)[l^{\pm}]$

Slicing tangles

into products of crossings G_i and horizontal pairs E_i

The generators G_1, G_2, E_1, E_2 of KaT_3

 ,  ,  , 

$BMW(\mathrm{A}_{n-1}) \cong KaT_n$

- surjective homomorphism $e_i \mapsto E_i$, $g_i \mapsto G_i$
- $\dim(BMW(\mathrm{A}_{n-1})) \leq n!!$

Defining relations $BMW(\mathrm{A}_{n-1})$:

(braid1)	$g_i g_j = g_j g_i$	when	$i \not\sim j$,
(braid2)	$g_i g_j g_i = g_j g_i g_j$	when	$i \sim j$,
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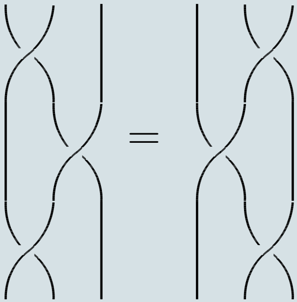
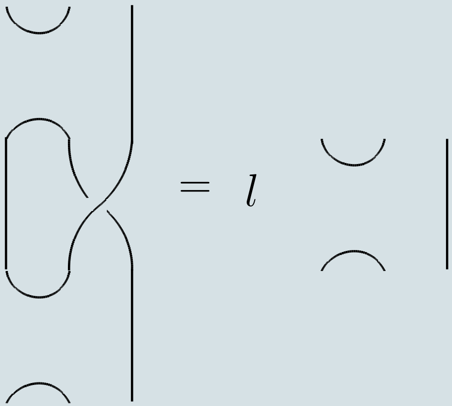
braid2



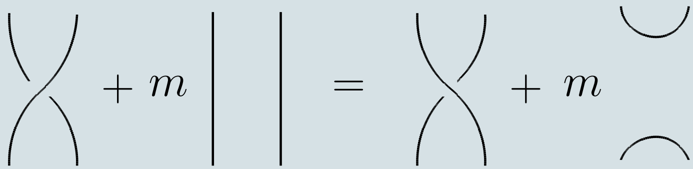
$g_1 g_2 g_1 = g_2 g_1 g_2$ holds by Reidemeister III

self-intersection2



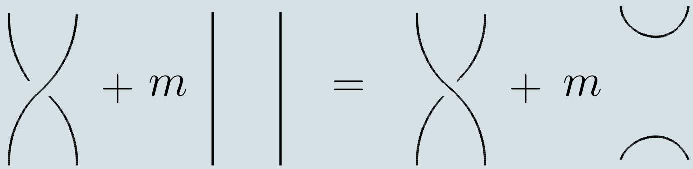
$$e_1 g_2 e_1 = l e_1$$

The skein relation



$$g_i + m = g_i^{-1} + m e_i$$

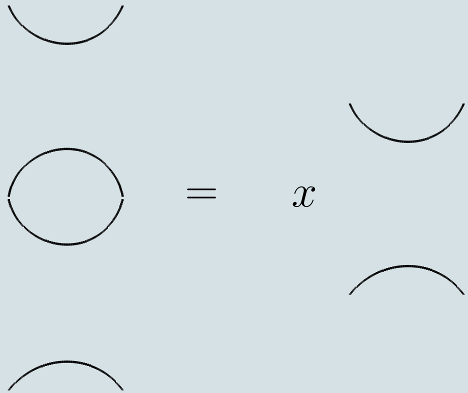
The skein relation



$$g_i + m = g_i^{-1} + m e_i$$

equivalently: $$e_i = l m^{-1}(g_i^2 + m g_i - 1)$$

The idempotent and the circle



$e_i^2 = x e_i$, so $x^{-1} e_i$ is an idempotent.

Brauer algebra

- A (n, k)-Brauer diagram is obtained from a tangle by forgetting whether a crossing is over or under
- An (n, k)-Brauer diagram is a matching of $\{1, \ldots, n + k\}$
- Coefficients from $R = \mathbb{Q}(x)[l^{\pm}]$ specialize $l \mapsto 1$, $m \mapsto 0$
- Brauer algebra is an algebra of dimension $n!!$ over $\mathbb{Q}(x)$
- $BMW(\mathrm{A}_{n-1})$ has dimension $n!!$ over $\mathbb{Q}(x, l)$

$n!! =$ the number of matchings of $2n$ points $= 1 \cdot 3 \cdots (2n - 1)$

$(1 - x)m = l - l^{-1}$

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$(1 - x)m = l - l^{-1}$

Left ideal $I_p = KaT_n e_1 e_3 \cdots e_{2p-1}$

- the p bottom strands are preserved
- more bottom strands may arise
- quotient out tangles with more than t bottom strands: I_p/I_{p+1}
- each irr repr of Sym_{n-2p} determines an irr repr of KaT_n via I_p/I_{p+1}
- by varying p, obtain all irreducible representations of KaT_n
- $p = 0$: $e_1 = 0$, the Hecke algebra of Sym_n
- $p = 1$, 1-dim repr of Sym_{n-2}: Krammer representation

Theorem

$BMW(\mathrm{A}_{n-1})$ is a sum of matrix algebras and has dimension

$$\sum_{p=0}^{\lfloor n/2 \rfloor} \left(\frac{n!}{2^p p! (n-2p)!} \right)^2 \cdot (n-2p)! = n!!$$

$n!!$ = the number of matchings of $2n$ points $= 1 \cdot 3 \cdots (2n-1)$

M is a graph

Artin group $A(M)$

generators $\{s_i \mid i \in M\}$

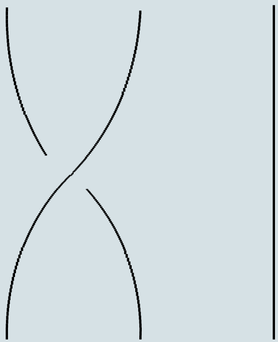
relations

(braid1) $s_i s_j = s_j s_i$ when $i \not\sim j$,

(braid2) $s_i s_j s_i = s_j s_i s_j$ when $i \sim j$

Theorem (Krammer, Zinno) $A(\mathrm{A}_{n-1})$ embeds in KaT_n via $s_i \mapsto G_i$.

Example: Mapping s_1, s_2 of $A(\mathrm{A}_2)$ to G_1, G_2 of KaT_3



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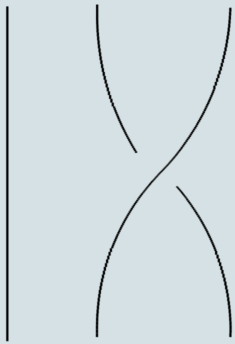
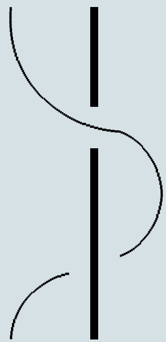


Intermezzo: links with type B_n

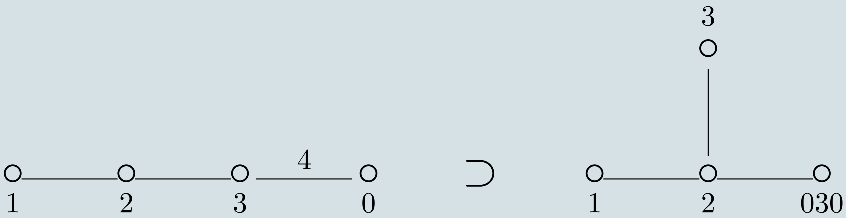
Affine BMW and type $M = \mathrm{B}_n$

- Replace knot space by solid torus
- Allcock: strands in presence of a pole
- Analog of KaT_n by Goodman & Hauschild

Extra generator G_0 crossing the pole



From B_n to D_n



$$(0^{-1}30)2(0^{-1}30) = 2(0^{-1}30)2$$

$$(030)3 = 3(030)$$

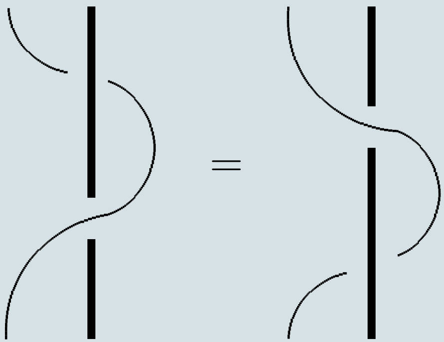
works when $0^2 = 1$

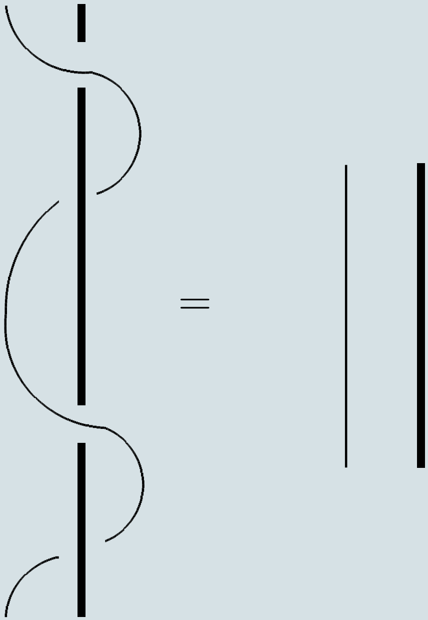
$\Rightarrow$ orbifold with a point of order 2

Towards a tangle realization of $BMW(\mathrm{D}_n)$

- Tangles live in orbifold with isolated point of order 2
- Allcock: Artin group in presence of a pole of order 2
- KaT_n^D variation of KaT_n with pole of order 2

G_0: twisting around a pole of order 2



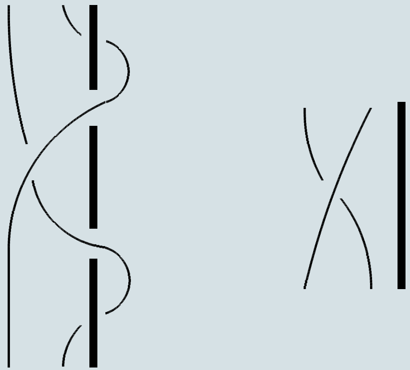


No Reidemeister rules for the pole

Goodman & Hauschild have two more relations.

Kauffman tangles for D_n

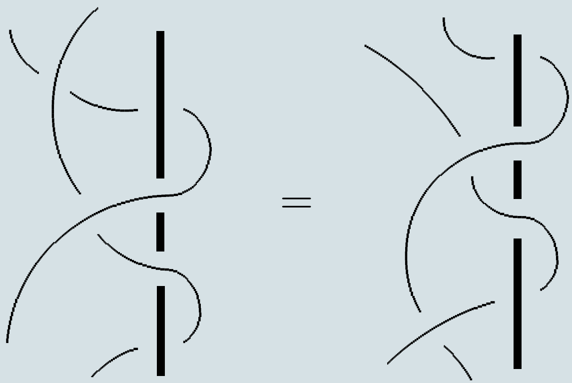
KaT_2^D generators:



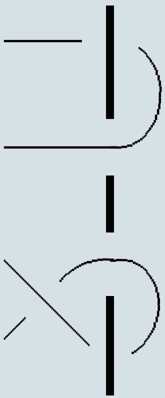
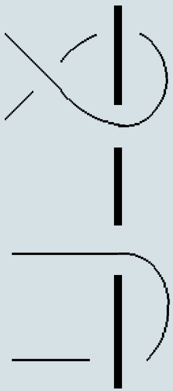
$G_1 = G_0 G_2 G_0$ G_2

So: number of pole twists is even.

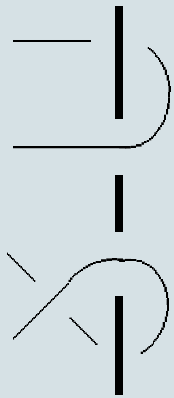
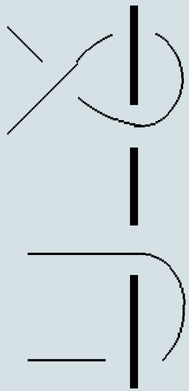
Braid relation with pole

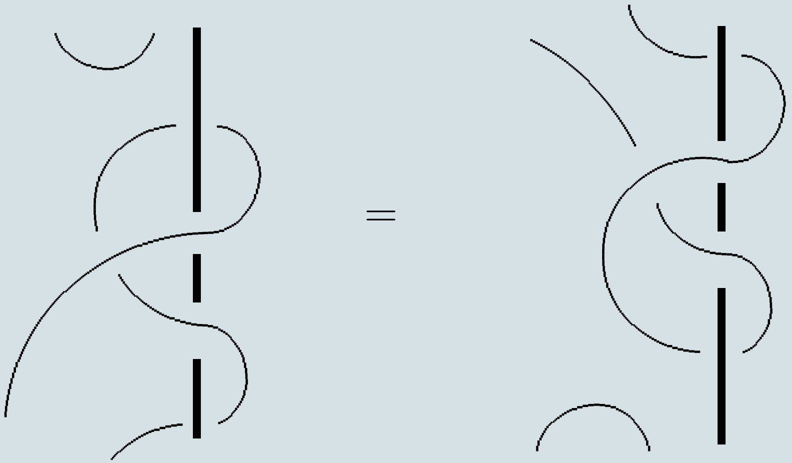


$$G_1G_2 = G_2G_1$$

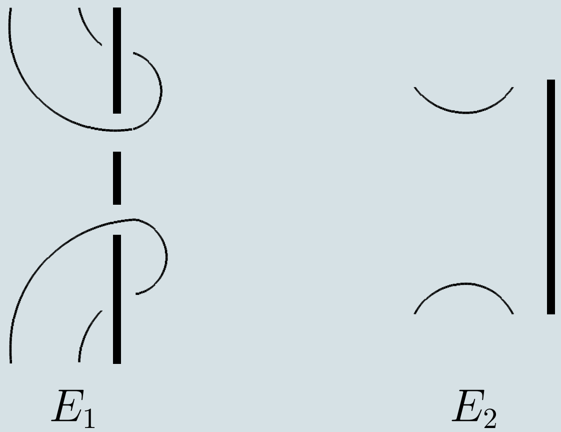
 = 

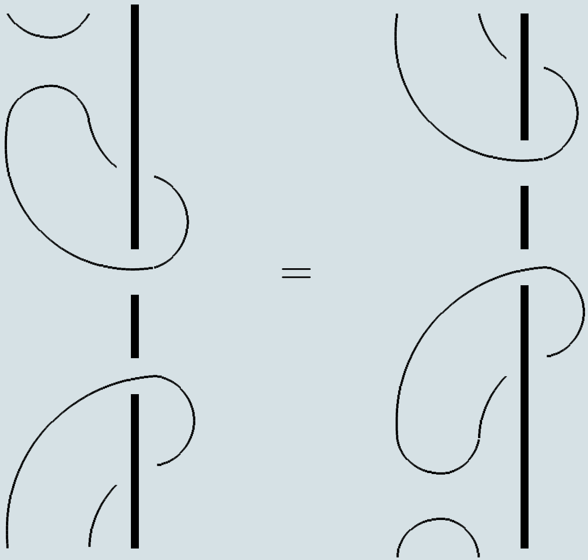
and

 = 

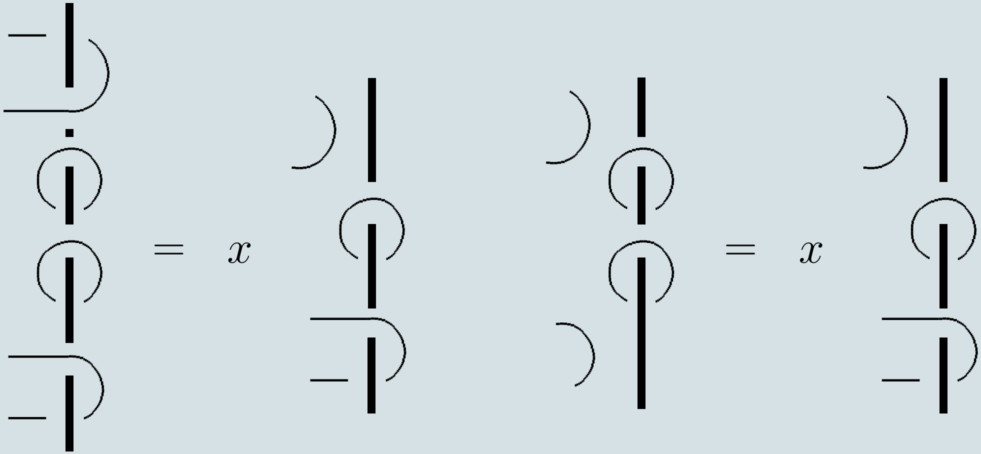


Horizontal strands





$$E_1 E_2 = E_2 E_1$$


= x
= x

,

Coefficient ring $KaT_0^D = R + R\Xi + R\Theta$

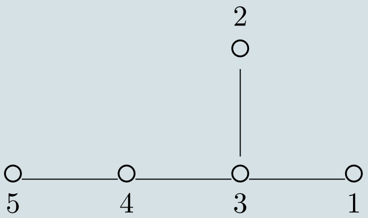
Generators

- $R = KaT_0$
- Ξ double twist with a self-intersection
- Θ two disjoint loops around the pole

Relations

- $\Xi^2 = x^2 - mx\Xi + ml^{-1}x\Theta$
- $\Theta^2 = x^2\Theta$
- $\Theta\Xi = \Xi\Theta = x^2l^{-1}\Theta$

Theorem $BMW(\mathrm{D}_n) \cong KaT_n^D$.



$g_i \mapsto G_i, \quad e_i \mapsto E_i,$

$e_1 g_2 e_1 = x g_2 e_1 \mapsto \Xi E_1, \quad e_1 e_2 e_1 = x e_1 e_2 \mapsto \Theta E_1$

Brauer algebra of type D_n

Decorations on strands to indicate pole twists.

- $2^{n-1}n!!$ without Ξ and Θ factor
- $2^{n-1}(n!! - n!)$ with a Ξ factor
- $n!! - n!$ with a Θ factor

Brauer algebra of type D_n

Decorations on strands to indicate pole twists.

- $2^{n-1}n!!$ without Ξ and Θ factor
  * even number of twists
- $2^{n-1}(n!! - n!)$ with a Ξ factor
- $n!! - n!$ with a Θ factor

Brauer algebra of type D_n

Decorations on strands to indicate pole twists.

- $2^{n-1}n!!$ without Ξ and Θ factor

- $2^{n-1}(n!! - n!)$ with a Ξ factor
  ** presence of a horizontal strand required
- $n!! - n!$ with a Θ factor

Brauer algebra of type D_n

Decorations on strands to indicate pole twists.

- $2^{n-1}n!!$ without Ξ and Θ factor
- $2^{n-1}(n!! - n!)$ with a Ξ factor
- $n!! - n!$ with a Θ factor

  *** all decorations vanish

Brauer algebra of type D_n

Decorations on strands to indicate pole twists.

- $2^{n-1}n!!$ without Ξ and Θ factor
- $2^{n-1}(n!! - n!)$ with a Ξ factor
- $n!! - n!$ with a Θ factor

Total $= (2^n + 1)n!! - (2^{n-1} + 1)n! = \dim(BMW(\mathrm{D}_n))$

Reps of $A(\mathrm{D}_n)$ in KaT_n^D

- Invariant bottom patterns are
  p horizontal strands, and the presence of Θ.
- For fixed p the dimension of the representation
  - with Θ is $\frac{n!}{2^p p!(n-2p)!}$ over Hecke(A_{n-2p-1})
    (if $n = 2p$ there are 2)
  - without Θ is $\frac{(n+1)!}{2^p p!(n-2p+1)!}$ over Hecke($\mathrm{A}_1\mathrm{D}_{n-2p}$)

Correspondence roots and horizontal strands

For A_{n-1}

- bottom strand from i to $j \leftrightarrow \epsilon_i - \epsilon_j$

For D_n

- bottom strand from i to j without Θ
  $\leftrightarrow \epsilon_i - \epsilon_j$ if not decorated
  $\leftrightarrow \epsilon_i + \epsilon_j$ if decorated
- bottom strand from i to j with $\Theta \leftrightarrow \epsilon_i \pm \epsilon_j$

Transition to roots

- Representation bases:
  $W(M)$-orbits of sets of mutually orthogonal positive roots
- Only admissible orbits occur:
  no reflection moves exactly 3 roots of any set in the orbit
- $\Rightarrow \mathrm{D}_n$ reps on $BMW(\mathrm{D}_n)e_1e_2$ factor through $BMW(\mathrm{A}_{n-1})$

Theorem* (M simply laced)

1. If $\mathcal{B}$ is admissible, then there is a subdiagram $C_{\mathcal{B}}$ of M and an irr repr of BMW over Hecke($C_{\mathcal{B}}$) with basis $\mathcal{B}$
2. $\dim(BMW) = \sum_{\mathcal{B} \text{ admissible}} |\mathcal{B}|^2 \, |W(C_{\mathcal{B}})|$

Properties of $BMW(M)$

- J coclique in M of size p
  $\Rightarrow$
  $e_J := x^{-p} \prod_{j \in J} e_i$ idempotent
- I_p := ideal generated by all e_J for $|J| = p$
- quotient by I_1 gives Hecke(M)
- the left module spanned by e_1 modulo I_2 is
  the Lawrence-Krammer representation
  with coefficients in Hecke algebra of type C
  where C is type of $C_{W(M)}(\alpha_1)$
- $\dim_C(BMW(M)e_J)$ = #($W(M)$-orbit of admissible closure of J)

BMW structure

M	$\|B\|$	$\|\mathcal{B}\|$	Y	C	$N_W(B)$
A_n	t	$\frac{(n+1)!}{2^t t!(n-2t+1)!}$	A_{n-2t}	A_{n-2t}	$2^t\,\mathrm{Sym}_t\,\mathrm{Sym}_{n+1-2t}$
D_n	t	$\frac{n!}{t!(n-2t)!}$	$A_1^t D_{n-2t}$	$A_1 D_{n-2t}$	$2^{2t}\,\mathrm{Sym}_t\,W(D_{n-2t})$
D_n	$2t$	$\frac{n!}{2^t t!(n-2t)!}$	D_{n-2t}	A_{n-2t-1}	$2^{2t} W(B_t) W(D_{n-2t})$
E_6	1	36	A_5	A_5	$2\,\mathrm{Sym}_6$
E_6	2	270	A_3	A_2	$2^{2+1}\,\mathrm{Sym}_4$
E_6	4	135	$\emptyset$	$\emptyset$	$2^4\,\mathrm{Sym}_4$
E_7	1	63	D_6	D_6	$2W(D_6)$
E_7	2	945	$A_1 D_4$	$A_1 D_4$	$2^{2+1+1} W(D_4)$
E_7	3	315	D_4	A_2	$2^3\,\mathrm{Sym}_3\,W(D_4)$
E_7	4	945	A_1^3	A_1	$2^{4+3}\,\mathrm{Sym}_4$
E_7	7	135	$\emptyset$	$\emptyset$	$2^7 L(3,2)$
E_8	1	120	E_7	E_7	$2W(E_7)$
E_8	2	3780	D_6	A_5	$2^{2+1} W(D_6)$
E_8	4	9450	D_4	A_2	$2^4\,\mathrm{Sym}_3\,W(D_4)$
E_8	8	2025	$\emptyset$	$\emptyset$	$2^{8+3} L(3,2)$

Y is root system orthogonal to B

C maximal subsystem of Y on nodes of M

Theorem*

$BMW(M)$ is finite-dimensional

$\Leftrightarrow$

connected components of M in $\{A_n, D_n, E_6, E_7, E_8 \mid n \in \mathbb{N}\}$.

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\dim BMW(E_6) &= 1,440,585 \\
\dim BMW(E_7) &= 439,670,025 \\
\dim BMW(E_8) &= 53,328,069,225
\end{aligned}
$$

$n!!$ = the number of matchings of $2n$ points $= 1 \cdot 3 \cdots (2n-1)$

Thanks