An Invitation to Tropical Geometry

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Outline

1. A-Discriminants Δ_A
2. Tropical Geometry
3. Tropical A-Discriminants
4. The Newton Polytope of Δ_A

This is joint work with Alicia Dickenstein and Bernd Sturmfels

arXiv:math.AG/0510126, J. Amer. Math. Soc., to appear.

1. Discriminants: Classical Examples

1. Discriminant of a quadratic polynomial in 1 variable

$$f(t) \;=\; x_2 t^2 + x_1 t + x_0\,, \qquad x_2 \neq 0$$

f has a double root $\iff$ $\Delta_f \;=\; x_1^2 - 4x_2x_0 = 0$

2. Discriminant of a cubic polynomial in 1 variable

$$f(t) \;=\; x_3 t^3 + x_2 t^2 + x_1 t + x_0, \;\; x_3 \neq 0$$

f has a double root $\iff$

$$\Delta_f \;=\; 27\,x_1^2x_4^2 - 18\,x_1x_2x_3x_4 + 4\,x_1x_3^3 + 4\,x_2^3x_4 - x_2^2x_3^2 = 0$$

A-Discriminants

[Gelfand, Kapranov, Zelevinsky 1992]

$A = \big(a_1 \cdots a_n\big) \in \mathbb{Z}^{d\times n}$, $(1,\ldots,1) \in$ row span A, $a_1,\ldots,a_n$ span $\mathbb{Z}^d$

A represents a family of hypersurfaces in $(\mathbb{C}^*)^d$ defined by

$$f_A(t) \;=\; \sum_{j=1}^{n} x_j\, t^{a_j} \;=\; \sum_{j=1}^{n} x_j\, t_1^{a_{1j}} t_2^{a_{2j}} \ldots t_d^{a_{dj}}\;.$$

$X_A^* \;=$ cl $\{(x_1 : \ldots : x_n) \in \mathbb{CP}^{n-1} \,|\, f_A(t) = 0$ has a singular point in $(\mathbb{C}^*)^d\}$

Generically, $\operatorname{codim} X_A^* \;=\; 1$, and

$$X_A^* \;=\; V(\Delta_A),$$

where Δ_A irreducible polynomial in $\mathbb{Z}[x_1,\ldots,x_n]$, the A-discriminant.

A-Discriminants: Classical Examples

1. Discriminant of a quadratic polynomial in 1 variable

$$f(t) = x_2 t^2 + x_1 t + x_0 \,, \qquad x_2 \neq 0 \qquad\qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

f has a double root $\iff$ $\Delta_A = x_1^2 - 4x_2x_0 = 0$

2. Discriminant of a cubic polynomial in 1 variable

$$f(t) = x_3 t^3 + x_2 t^2 + x_1 t + x_0, \;\; x_3 \neq 0 \qquad\qquad A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

f has a double root $\iff$

$$\Delta_A = 27\, x_1^2 x_4^2 - 18\, x_1 x_2 x_3 x_3 + 4\, x_1 x_3^3 + 4\, x_2^3 x_4 - x_2^2 x_3^2 = 0$$

A-Discriminants: Classical Examples

3. Resultant of two polynomials in 1 variable

$$f(t) = \sum_{i=0}^{n} x_i t^i, \quad x_n \neq 0, \qquad g(t) = \sum_{i=0}^{m} y_i t^i, \quad y_m \neq 0,$$

f and g have a common root $\iff$ $\mathrm{Res}(f,g) = 0$

$\mathrm{Res}(f,g) = \Delta_A \in \mathbb{Z}[x_0,\ldots,x_n,y_0,\ldots,y_m]$ for

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & n & 0 & 1 & \ldots & m \end{pmatrix}$$

$\mathrm{Res}(f,g) =$ determinant of the Sylvester matrix

A-Discriminants: More Examples

4. Discriminant of a deg 2 homogeneous polynomial in 3 variables

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

$$\Delta_A = 1/2 \det \begin{pmatrix} 2x_1 & x_2 & x_4 \\ x_2 & 2x_3 & x_5 \\ x_4 & x_5 & 2x_6 \end{pmatrix}$$

5. Discriminant of a deg 3 homogeneous polynomial in 3 variables

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 0 & 1 & 0 \end{pmatrix}$$

$\deg \Delta_A = 12$, $\quad 2040$ terms

Newton Polytopes

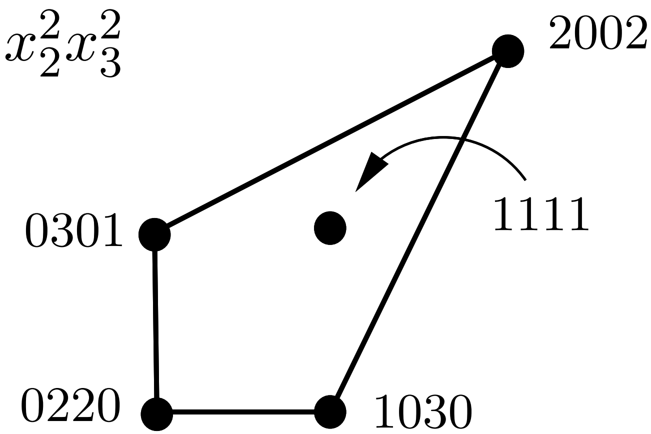
$$g \;=\; \sum_{c \in C} \gamma_c \, x^c \;=\; \sum_{c \in C} \gamma_c \; x_1^{c_1} \cdots x_n^{c_n} \, , \qquad \gamma_c \in \mathbb{C}^*, \; C \subset \mathbb{Z}^n$$

$$\textcolor{red}{\mathrm{New}(g)} \;=\; \mathrm{conv}\,\{c \,|\, c \in C\} \;\subseteq\; \mathbb{R}^n$$

Newton polytope

Example:

$$g \;=\; 27\, x_1^2 x_4^2 - 18\, x_1 x_2 x_3 x_4 + 4\, x_1 x_3^3 + 4\, x_2^3 x_4 - x_2^2 x_3^2$$



Once we know $\mathrm{New}(\Delta_A)$, determining Δ_A is merely a linear algebra problem!

A-Discriminants: Our Goals

Goal:

Derive information on Δ_A, resp. X_A^*, for instance

- $\deg \Delta_A$
- Newton polytope of Δ_A

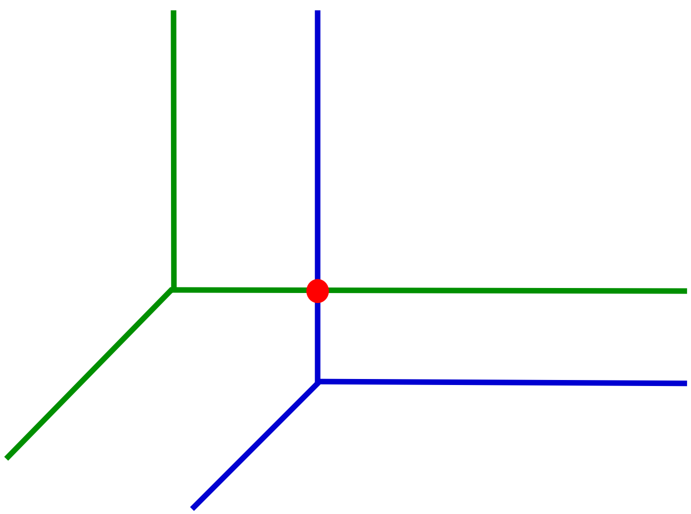
directly from the matrix, i.e., the point configuration A.

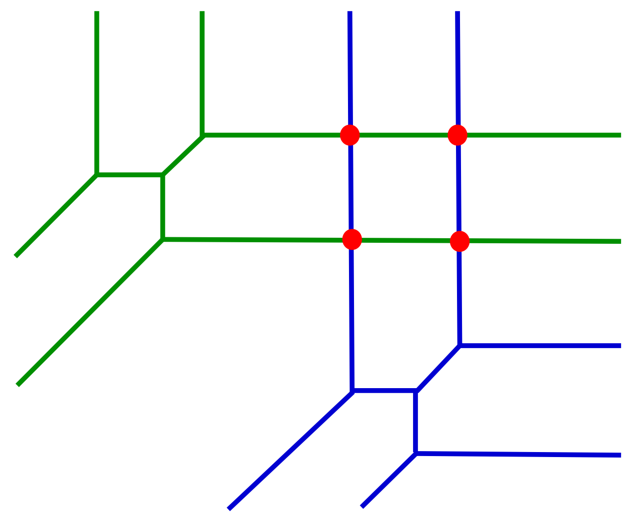
Ansatz: Study the tropicalization of X_A^* !

2. Tropical Geometry

Tropical geometry is algebraic geometry over the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$, $\qquad x \oplus y := \min\{x, y\}$, $x \otimes y := x + y$.

algebraic varieties $\overset{\tau}{\longrightarrow}$ tropical varieties, i.e. polyhedral fans





Tropical Varieties – the Algebraic Approach

$Y \subseteq \mathbb{CP}^{n-1}$ irreducible variety, $\dim Y = r$,
$I_Y \subseteq \mathbb{C}[x_1, \ldots, x_n]$ defining prime ideal.

For $w \in \mathbb{R}^n$ and $f = \sum_{c \in C} \gamma_c x^c$, $\gamma_c \in \mathbb{C}$, $C \subset \mathbb{Z}^n$, define

$$\mathrm{in}_w f = \sum_{w \cdot c \min} \gamma_c x^c$$ initial term of f,

$$\mathrm{in}_w(I_Y) = \langle \mathrm{in}_w f \mid f \in I_Y \rangle$$ initial ideal of I_Y.

$$\tau(Y) = \{ w \in \mathbb{R}^n \mid \mathrm{in}_w(I_Y) \text{ does not contain a monomial} \}$$

tropicalization of Y

$\tau(Y)$ is a pure r-dimensional polyhedral fan in $\mathbb{R}^n$, resp. $\mathbb{TP}^{n-1}$.

Examples of Tropicalized Varieties

1. The discriminant of a cubic polynomial in 1 variable

$$\Delta \;=\; 27\,x_1^2x_4^2 - 18\,x_1x_2x_3x_4 + 4\,x_1x_3^3 + 4\,x_2^3x_4 - x_2^2x_3^2$$

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$\mathrm{in}_{(1,0,1,0)}(\Delta) \;=$

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$\mathrm{in}_{(1,0,1,0)}(\Delta) \;=\; 4x_2^3x_4$

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$\mathrm{in}_{(-1,-1,-1,0)}(\Delta) = 4x_1x_3^3 - x_2^2x_3^2$ $\quad (-1,-1,-1,0) \in \tau(X_A^*)$

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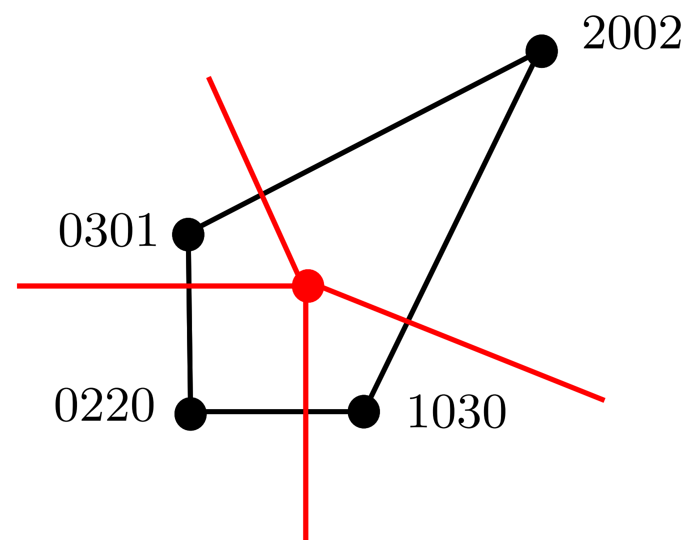


Examples of Tropicalized Varieties

2. Y hypersurface in $\mathbb{CP}^{n-1}$

$f \in \mathbb{C}[x_1, \ldots, x_n]$ irreducible polynomial defining Y
$\mathrm{New}(f)$ Newton polytope, $\mathcal{N}_{\mathrm{New}(f)}$ its normal fan

$$\tau(Y) \;=\; \text{codim } 1\text{-skeleton of } \mathcal{N}_{\mathrm{New}(f)}$$

Proof:

$$\begin{aligned}
\tau(Y) &= \{\, w \in \mathbb{R}^n \,|\, \mathrm{in}_w(f) \text{ is not a monomial} \,\} \\
&= \{\, w \in \mathbb{R}^n \,|\, \dim \big(\mathrm{New}(\mathrm{in}_w(f))\big) > 0\} \\
&= \{\, w \in \mathbb{R}^n \,|\, \dim \big(w\text{-minimal face of } \mathrm{New}(f)\big) > 0\} \\
&= \bigcup_{\substack{\sigma \in \mathcal{N}_{\mathrm{New}(f)} \\ \mathrm{codim}\, \sigma > 0}} \sigma
\end{aligned}$$

Examples of Tropicalized Varieties

3. $Y = X_A$ toric variety

$A \in \mathbb{Z}^{d\times n}$, X_A toric variety associated with conv$\{a_1, \ldots, a_n\}$.

$$\tau(Y) \;=\; \text{row span } A$$

Proof:

$$I_{X_A} \;=\; \langle\, x^u - x^v \,|\, u, v \in \mathbb{N}^n \text{ with } Au = Av \rangle$$

$$\begin{aligned} \tau(Y) &= \{\, w \in \mathbb{R}^n \,|\, \mathrm{in}_w(f) \text{ is not a monomial for any } f \in I_{X_A} \} \\ &= \{\, w \in \mathbb{R}^n \,|\, wu = wv \text{ whenever } Au = Av \} \\ &= \text{row span } A \end{aligned}$$

4. $Y = V$ linear, resp. projective subspace

$$\tau(Y) \;=\; \mathcal{B}(M(V))$$

Bergman fan of the matroid associated with V

Bergman Fans

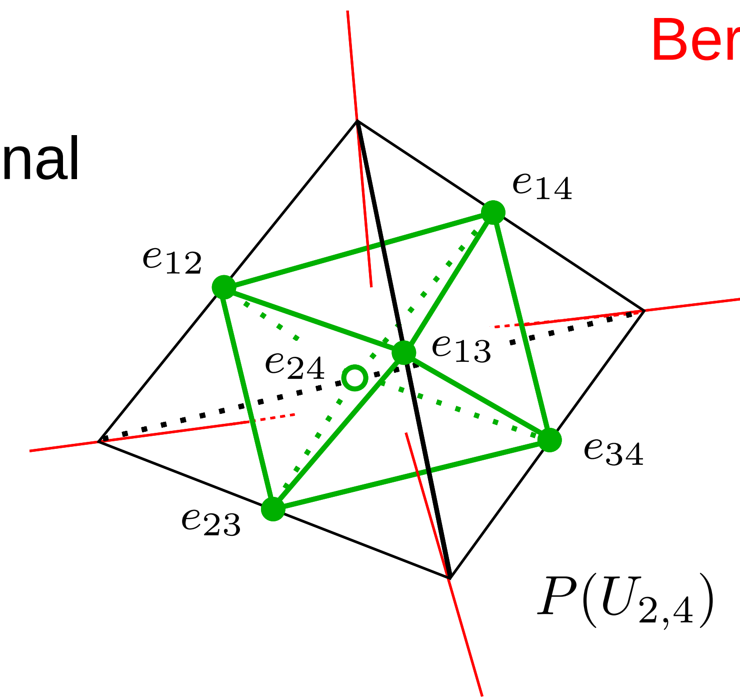
M matroid on $\{1,\ldots,n\}$, $\operatorname{rk} M = r$, $M \subseteq \binom{n}{r}$

$P(M) \; = \; \operatorname{conv}\{e_\sigma \,|\, \sigma \in M\}$, $e_\sigma \; = \; \sum_{i\in\sigma} e_i$ matroid polytope

$\mathcal{B}(M) \;\; = \;\; \{w \in \mathbb{R}^n \,|\, w$ – maximal face of $P(M)$ is the polytope of a loop-free matroid$\}$

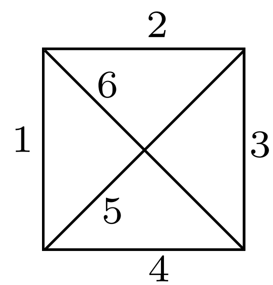
Bergman fan

$\mathcal{B}(M)$ is a $(\operatorname{rk} M - 1)$-dimensional subfan of $\mathcal{N}_{P(M)}$.

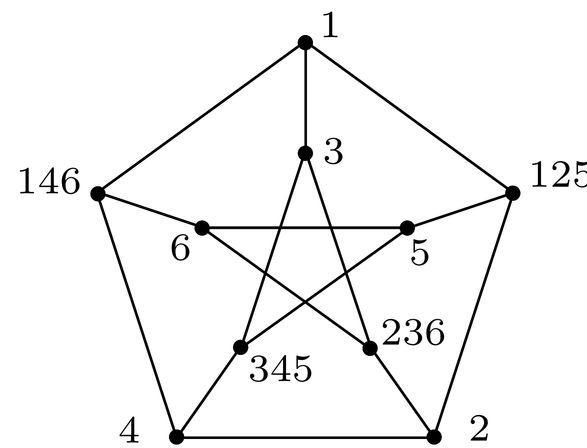


Examples of Bergman Fans

$M = M(K_4)$
$r = 3$, $n = 6$

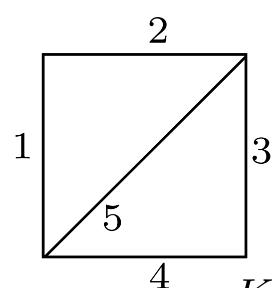


K_4

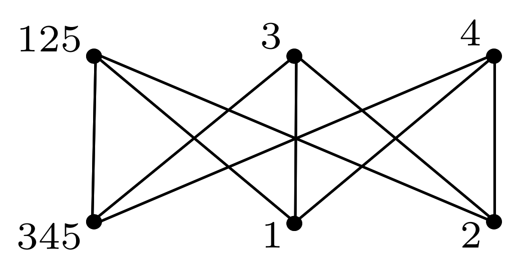


$B(M(K_4))$

$M = M(K_4 \backslash e)$
$r = 3$, $n = 5$



$K_4 \backslash e$



$B(M(K_4 \backslash e))$

Bergman Fans and Tropical Linear Spaces

$\mathcal{C}$ set of circuits of a matroid M on $\{1, \ldots, n\}$

$$\mathcal{B}(M) \;=\; \{w \in \mathbb{R}^n \,|\, \min\{w_j \,|\, j \in C\} \text{ is attained at least twice for any } C \in \mathcal{C}\}$$

4. $Y = V$ linear, resp. projective subspace

$$\tau(Y) \;=\; \mathcal{B}(M(V))$$

Proof:

$I_Y = \langle f_1, \ldots, f_t \rangle$, f_i linear forms in n variables

$\mathcal{C} \;=\; \{$ variables occurring in $f_i \mid i = 1, \ldots, t\}$

$$\begin{aligned}\tau(Y) \;&=\; \{w \in \mathbb{R}^n \,|\, \mathrm{in}_w(f_i) \text{ is not a monomial for any } i\} \\ &=\; \{w \in \mathbb{R}^n \,|\, \min\{w_j \,|\, j \in C\} \text{ is attained at least twice for any } C \in \mathcal{C}\}.\end{aligned}$$

Tropical Varieties – via Valuations

$K = \mathbb{C}\{\{t\}\}$ field of Puiseux series

$$\begin{array}{rccc} \mathrm{val}\ : & K^* & \longrightarrow & \mathbb{Q} \\ & \sum_{q\in\mathbb{Q}} a_q t^q & \longmapsto & \inf\{q\,|\,a_q \neq 0\,\} \end{array}$$

valuation

$$\mathrm{val}\ : \quad (K^*)^n \to \mathbb{Q}^n \hookrightarrow \mathbb{R}^n$$

Theorem: Let I be an ideal in $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, $V_{\mathbb{C}^*}(I)$, $V_{K^*}(I)$ the varieties of I over $\mathbb{C}^*$ and K^*, respectively. Then $\tau(V_{\mathbb{C}^*}(I))$ equals the closure of the image of $V_{K^*}(I)$ under val,

$$\tau(V_{\mathbb{C}^*}(I)) \;=\; \overline{\mathrm{val}(V_{K^*}(I))}\,.$$

3. Tropical A-Discriminants

$A = (a_1 \cdots a_n) \in \mathbb{Z}^{d \times n}$, $(1, \ldots, 1) \in$ row span A, $a_1, \ldots, a_n$ span $\mathbb{Z}^d$

Horn uniformization of A-discriminants: *[Kapranov '91]*

The variety X_A^* is the closure of the image of the morphism

$$\begin{array}{rccl} \varphi_A : & \mathbb{P}(\ker A) \times (\mathbb{C}^*)^d/\mathbb{C}^* & \longrightarrow & (\mathbb{CP}^{n-1})^* \\ & (u, t) & \longmapsto & (u_1 t^{a_1} : u_2 t^{a_2} : \cdots : u_n t^{a_n}) . \end{array}$$

Tropical Horn uniformization:

$$\begin{array}{rccl} \tau(\varphi_A) : & \mathcal{B}(\ker A) \times \mathbb{R}^d & \longrightarrow & \mathbb{TP}^{n-1} \\ & (w, v) & \longmapsto & w + vA \end{array}$$

$\operatorname{im} \tau(\varphi_A) = \mathcal{B}(\ker A) +$ row span A Horn fan

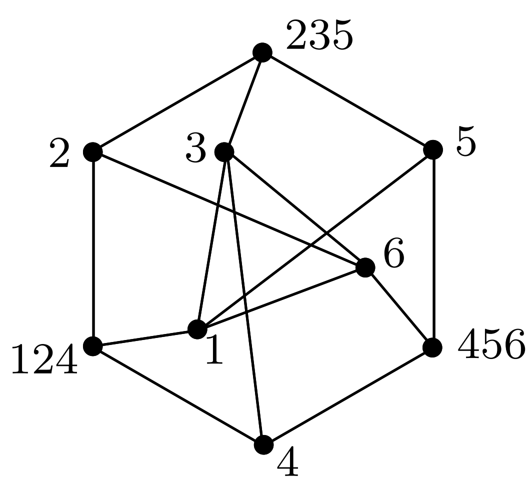
Tropical A-Discriminants

Theorem: *[DFS]*

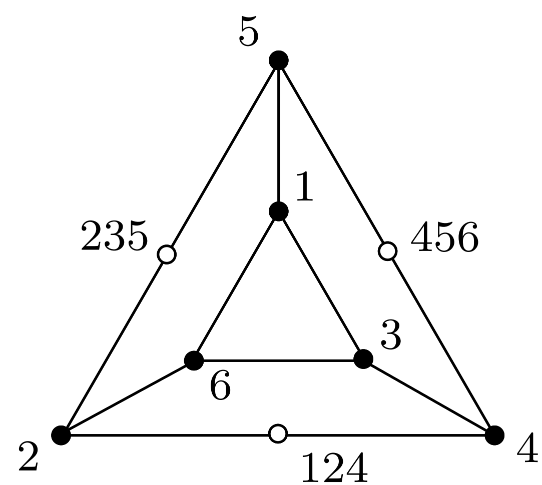
$$\tau(X_A^*) \;=\; \mathcal{B}(\ker A) + \text{row span } A$$

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$



$\mathcal{B}(\ker A)$

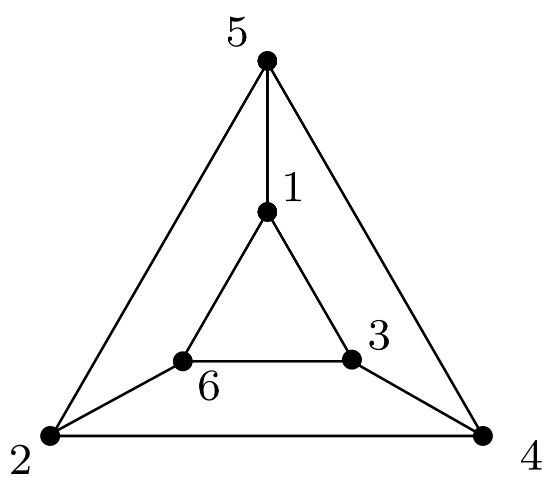


$\tau(X_A^*)$

4. The Newton Polytope of Δ_A

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$



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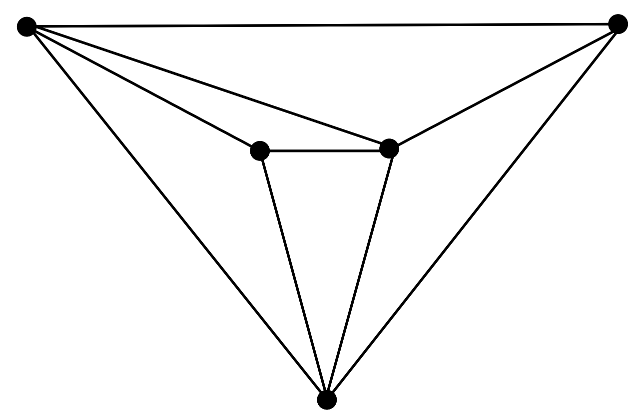

$\mathrm{New}(\Delta_A)$

The Newton Polytope of Δ_A

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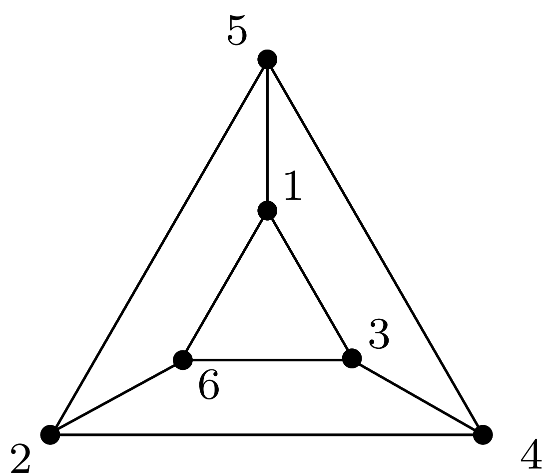

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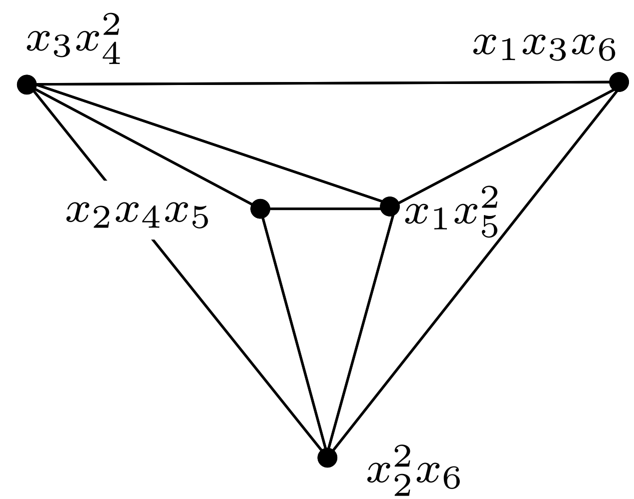

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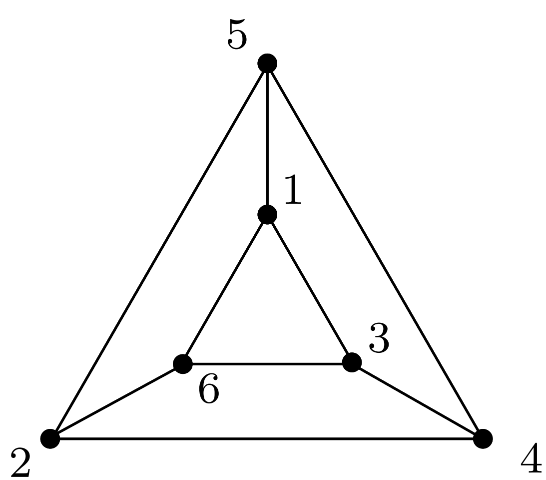

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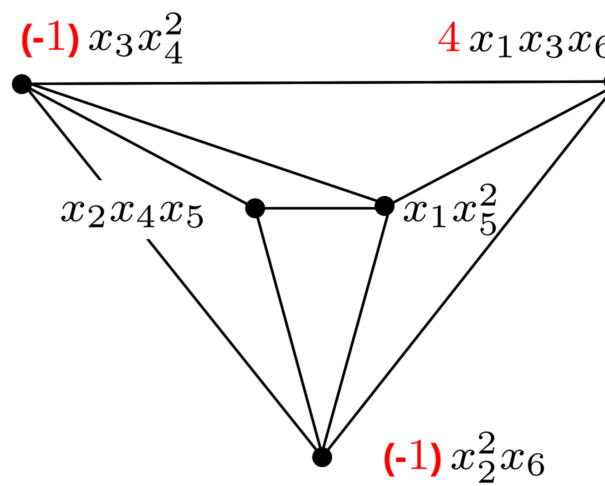

$\mathrm{New}(\Delta_A)$

The Newton Polytope of Δ_A

$A \in \mathbb{Z}^{d \times n}$, codim $X_A^* = 1$, $w \in \mathbb{R}^n$ generic.

Theorem: *[DFS]*
The exponent of x_i in the initial monomial $\mathrm{in}_w(\Delta_A)$ equals the number of intersection points of the halfray

$$w + \mathbb{R}_{>0} e_i$$

with the tropical discriminant $\tau(X_A^*)$, counting multiplicities:

$$\deg_{x_i}\big(\mathrm{in}_w(\Delta_A)\big) \;=\; \sum_{\sigma \in \mathcal{B}(\mathrm{ker A})_{i,w}} \big|\, \det\big(A^T, \sigma_1, \ldots, \sigma_{n-d-1}, e_i\big)\,\big|\,.$$

The Newton Polytope of Δ_A

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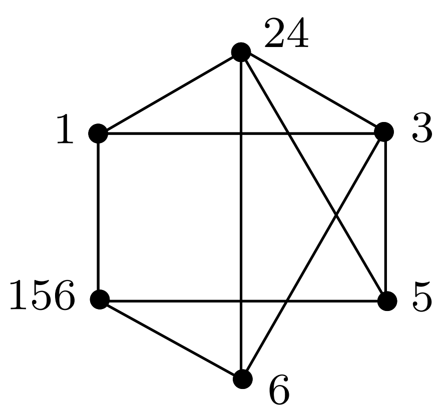
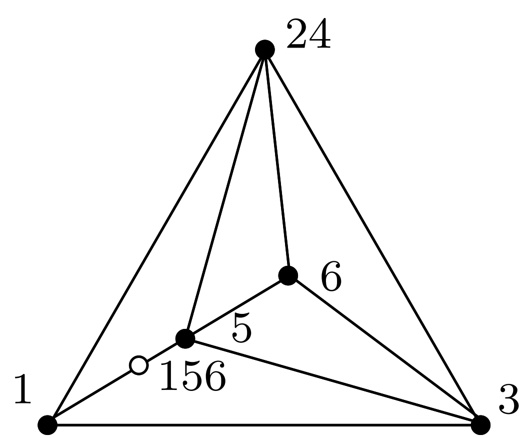
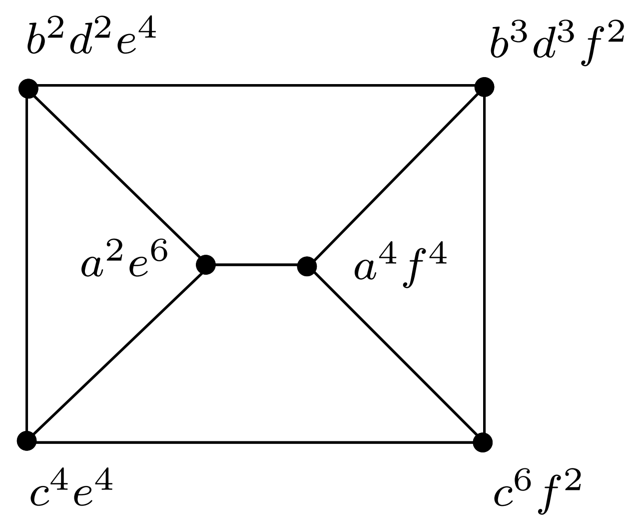


$\mathcal{B}(\ker A)$



$\tau(X_A^*)$



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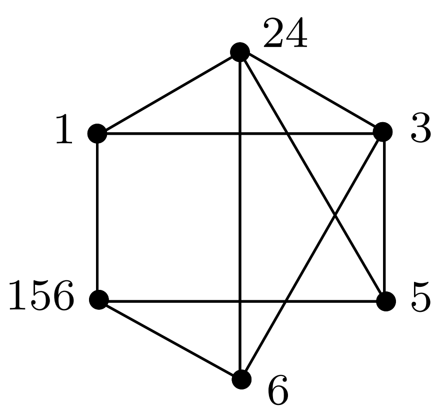
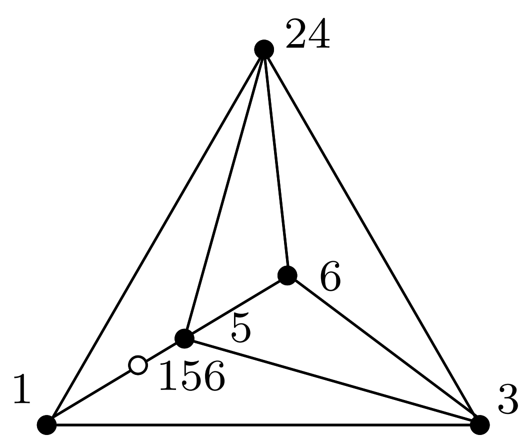
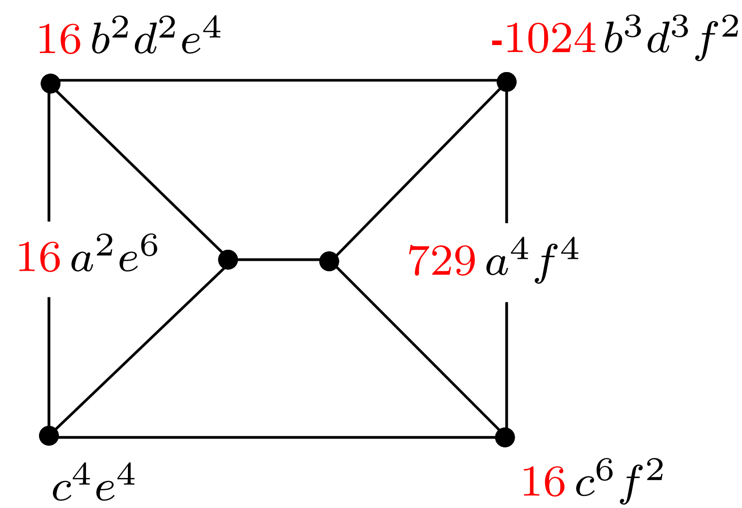


$\mathcal{B}(\ker A)$



$\tau(X_A^*)$



$\mathrm{New}(\Delta_A)$

The Newton Polytope of Δ_A

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{aligned} \Delta_A \;=\; & c^4e^4 - 8bc^2de^4 + 16b^2d^2e^4 - 8ac^2e^5 - 32abde^5 + 16a^2e^6 \\ & -8c^5e^2f + 64bc^3de^2f - 128b^2cd^2e^2f + 68ac^3e^3f \\ & +240abcde^3f - 144a^2ce^4f + 16c^6f^2 - 192bc^4df^2 \\ & +768b^2c^2d^2f^2 - 1024b^3d^3f^2 - 144ac^4ef^2 + 2304ab^2d^2ef^2 \\ & +270a^2c^2e^2f^2 - 1512a^2bde^2f^2 + 216a^3e^3f^2 + 216a^2c^3f^3 \\ & +2592a^2bcdf^3 - 972a^3cef^3 + 729a^4f^4 \end{aligned}$$

Summary and Outlook

Tropical Geometry

- allows for a new, constructive approach to A-discriminants, independent of any smoothness assumptions.
- opens the discrete-geometric toolbox for classical problems in algebraic geometry.
- establishes itself as a field on its own right on the border line of algebra, geometry and discrete mathematics.

Upcoming Event

MSRI program on Tropical Geometry

Fall 2009

E.M.F., Ilia Itenberg, Grigory Mikhalkin, Bernd Sturmfels

Please keep checking `www.msri.org` !

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