Inapproximability of the Tutte polynomial

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The Tutte polynomial (traditional bivariate style)

The *Tutte polynomial* of a graph $G = (V, E)$ is a two-variable polynomial T defined by

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{\kappa(A)-\kappa(E)} (y-1)^{|A|+\kappa(A)-n},$$

where $\kappa(A)$ denotes the number of connected components of (V, A).

Evaluations of the Tutte polynomial at various points and along various curves in $\mathbb{R}^2$ yield much interesting information about G.

NB. The Tutte polynomial is well defined for all matroids, but we consider only graphs (graphic matroids) today.

Evaluations of the Tutte polynomial

- $T(G;1,1)$ counts spanning trees in G.
- $T(G;2,1)$ counts forests in G.
- $T(G;1-q,0)$ counts q-colourings of G.
- More generally, along the hyperbola

$$H_q = \{(x,y) : (x-1)(y-1) = q\},$$

  $T(G;x,y)$ specialises to the partition function of the q-state Potts model.
- $T(G;2,0)$ counts acyclic orientations of G.
- Along the $y > 1$ branch of H_0, $T(G;1,y)$ specialises to the reliability polynomial of G.

Potts model

For q a positive integer,

$$Z_{\text{Potts}}(G; q, y) = \sum_{\sigma: V \to [q]} y^{|\text{mono}(\sigma)|},$$

where

$$\text{mono}(\sigma) = \{\{i, j\} \in E : \sigma(i) = \sigma(j)\},$$

is the partition function of the q-state Potts model.

- $y > 1$: Ferromagnetic, favours like spins.
- $y < 1$: Antiferromagnetic, favours unlike spins.

Tutte vs Potts

Fact

When $q = (x-1)(y-1)$ is a positive integer,

$$T(G; x, y) = \text{normalising factor} \times Z_{\text{Potts}}(G; q, y)$$

Proof.

E.g., use "Recipe Theorem":

$$Z_{\text{Potts}}(G; q, y) = (y-1)\, Z_{\text{Potts}}(G/e; q, y) + Z_{\text{Potts}}(G \setminus e; q, y).$$

□

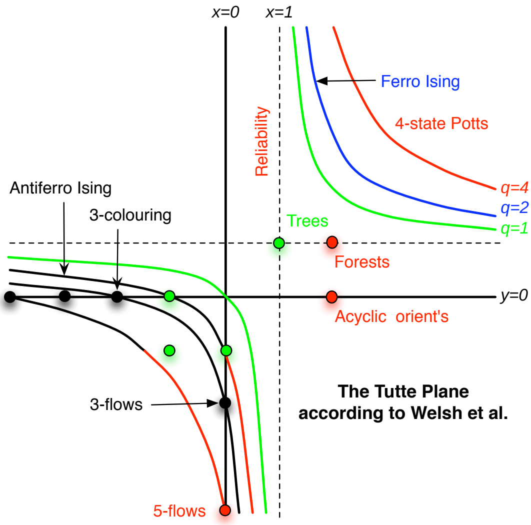
Mapping the Tutte plane: exact evaluation

Recall H_q is the hyperbola $H_q = \{(x, y) : (x-1)(y-1) = q\}$.

- The hyperbola H_1 is trivial.
- The special points $(1, 1)$, $(-1, 0)$, $(0, -1)$ and $(-1, -1)$ are all polynomial-time.
- All other points are #P-hard.

I have restricted attention to points in the real plane, but the extension to complex points has been completely worked out [Jaeger, Vertigan and Welsh, 1990].

The Tutte plane



Approximate computation: FPRAS

Definition

An *FPRAS* is a randomised algorithm that produces a result that is correct to within relative error $1 \pm \varepsilon$ with high probability. It must run in time poly(n, ε^{-1}), where n is the input size.

Goal: Map the regions/curves/points of the Tutte plane that admit an FPRAS.

The ferromagnetic Ising model

The positive branch of the hyperbola H_2 corresponds to the ferromagnetic Ising model (with no applied field).

Theorem (J. & Sinclair)

The positive branch of H_2 admits an FPRAS.

In fact, it is even possible to handle a non-zero applied field, provided the field acts in the same sense at every vertex.

Inapproximability of the Tutte polynomial

Observation

If $T(G; x, y)$ has a combinatorial interpretation as counting certain structures, and the decision problem for those structures is NP-complete, then there is no FPRAS for $G \mapsto T(G; x, y)$ unless $\mathrm{RP} = \mathrm{NP}$.

Proof.

An FPRAS must in particular distinguish between zero and non-zero. □

Example

The point $(x, y) = (-5, 0)$, since $T(G; -5, 0)$ counts 6-colourings of G.

Generalising a little

Lemma

Suppose $(x, y) \in H_6$, *i.e,* $(x-1)(y-1) = 6$, *and* $-1 < y < 1$. *Then there is no FPRAS for* $G \mapsto T(G; x, y)$ *unless* $\mathrm{RP} = \mathrm{NP}$.

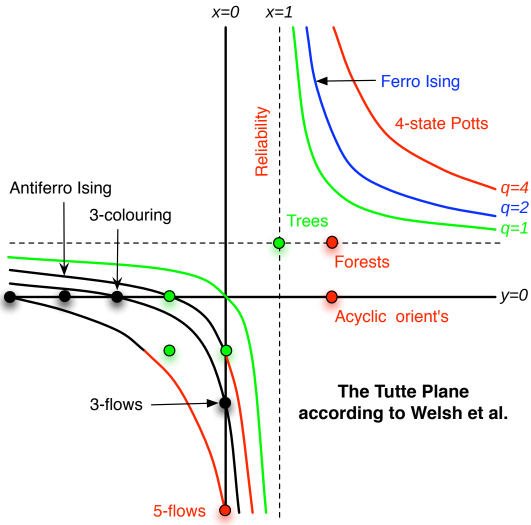
Proof.

Use a certain operation on graphs, called a "thickening", we may reduce $T(\cdot\,; -5, 0)$ to $T(\cdot\,; x, y)$. □

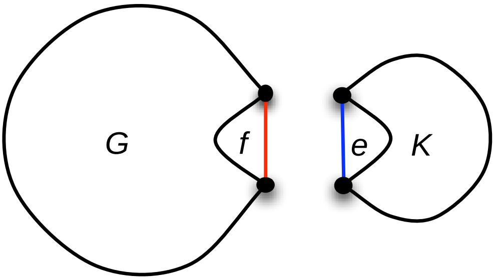
Remark

This approach does not let us deal with the point $(0, -5)$. Note that $T(G; 0, -5)$ counts *nowhere-zero 6-flows* in G. Seymour has shown that $T(G; 0, -5) > 0$ iff G is bridgeless, so the decision problem is trivial!

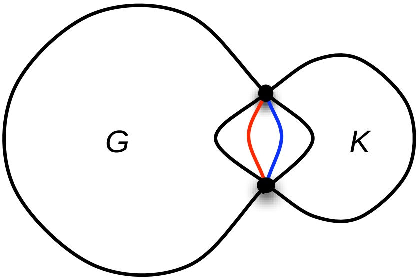
The Tutte plane (reprise)



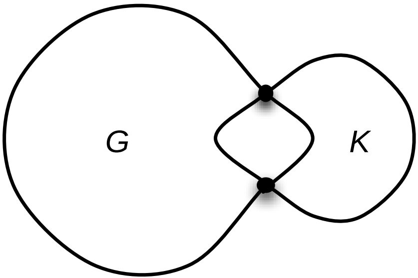
2-sums and tensor products



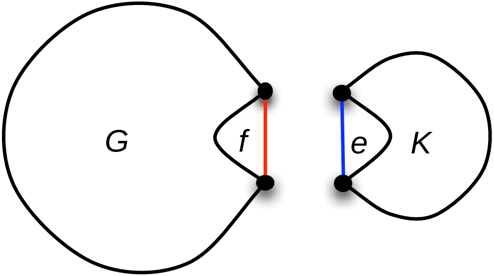
2-sums and tensor products



2-sums and tensor products

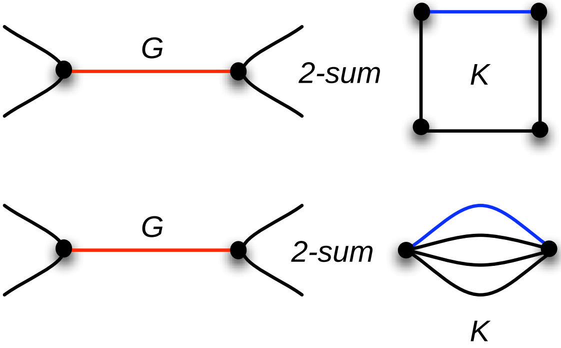


2-sums and tensor products

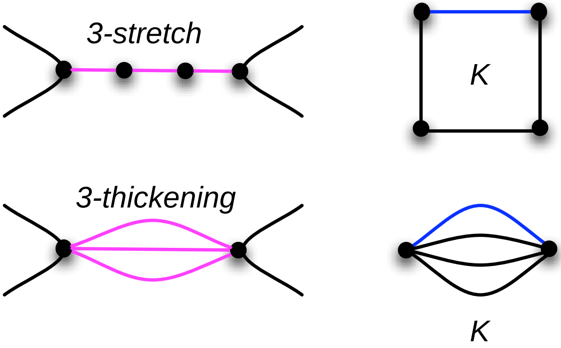


The tensor product $G \otimes K$ of G and K is got by taking a 2-sum of G and K along every edge f of G.

Stretchings and thickenings



Stretchings and thickenings



Shifts

Lemma

$T(G \otimes K; x, y) = A\,T(G; x', y')$, where x' and y' depend on x, y and K, and A is easily computable.

Definition

If (x, y) and (x', y') are in the above relation then we say that (x, y) may be *shifted to* (x', y').

$$(x', y') = \begin{cases} (x^k,\ q/(x^k - 1) + 1) & \text{for a } k\text{-stretch;} \\ (q/(y^k - 1) + 1,\ y^k) & \text{for a } k\text{-thickening.} \end{cases}$$

Moving now to work in progress [Goldberg & J., 2006]. . .

Key lemmas

Lemma

Suppose $(x, y) \in \mathbb{Q}^2$ *satisfies* $q = (x-1)(y-1) \notin \{1, 2\}$. *Suppose also that it is possible to shift the point* (x, y) *to the point* (x', y') *with* $y' \notin [-1, 1]$, *and to* (x'', y'') *with* $y'' \in (-1, 1)$. *Then there is no FPRAS for the function* $G \mapsto T(G; x, y)$ *unless* $\mathrm{RP} = \mathrm{NP}$.

Lemma

Suppose $(x, y) \in \mathbb{Q}^2$ *satisfies* $q = (x-1)(y-1) \notin \{0, 1, 2\}$. *Suppose also that it is possible to shift the point* (x, y) *to the point* (x', y') *with* $x' \notin [-1, 1]$, *and to* (x'', y'') *with* $x'' \in (-1, 1)$. *Then there is no FPRAS for the function* $G \mapsto T(G; x, y)$ *unless* $\mathrm{RP} = \mathrm{NP}$.

Example

$$
\begin{aligned}
(x,y) = \Big(-\frac{1}{3}, 0\Big) &\mapsto \Big(\frac{1}{9}, -\frac{1}{2}\Big) && \text{(2-stretch)} \\
&\mapsto \Big(-\frac{7}{9}, \frac{1}{4}\Big) && \text{(2-thickening)} \\
&\mapsto \Big(\frac{49}{81}, -\frac{19}{8}\Big) = (x', y') && \text{(2-stretch).}
\end{aligned}
$$

Note that $y \in (-1,1)$ and $y' \notin [-1,1]$.

So no FPRAS for the Tutte polynomial at $(-\frac{1}{3}, 0)$ unless RP = NP.

Sketch proof of Lemma 7

Multivariate Tutte polynomial [Sokal, 2005]:

$$T(G;x,y) = \sum_{A\subseteq E} (x-1)^{\kappa(A)-\kappa(E)} (y-1)^{|A|+\kappa(A)-n}$$
$$= C \sum_{A\subseteq E} (y-1)^{|A|} q^{\kappa(A)},$$

where $C = (x-1)^{-\kappa(E)}(y-1)^{-n}$ and $q = (x-1)(y-1)$.
In this form, we may assign a different weights to each edge:

$$\sum_{A\subseteq E} q^{\kappa(A)} \prod_{e\in A} w(e),$$

where $w : E \to \mathbb{R}$ is an edge weighting.

Sketch proof of Lemma 7 (continued)

The decision problem Minimum 3-way Cut is:

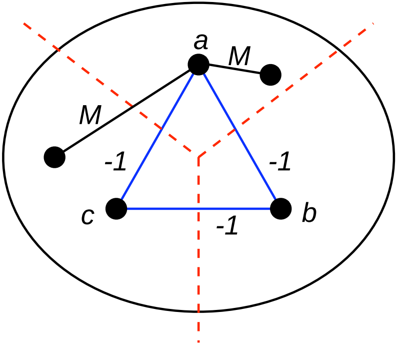
Instance A graph $G = (V, E)$ with three distinguished vertices ("terminals") $a, b, c \in V$, and an integer bound B.

Output Is there a set of at most B edges whose removal from G leaves a, b and c in distinct components?

It was shown to be NP-complete by Dahlhaus, Johnson, Papadimitriou, Seymour and Yannakakis.

The reduction of an instance of Minimum 3-way Cut to an instance of multivariate Tutte evaluation is easy to describe. . .

Sketch proof of Lemma 7 (continued)



- Three new -1 edges force a, b and c to lie in different components of the subgraph A.
- The original edges of G have weight M (large), maximising $|A|$.

Sketch proof of Lemma 7 (concluded)

The two shifts guaranteed in the statement of the lemma allow us to simulate edges of weight (very close to) -1, and also edges of arbitrarily large weight.

This completes the reduction.

The Tutte plane as at February 2006