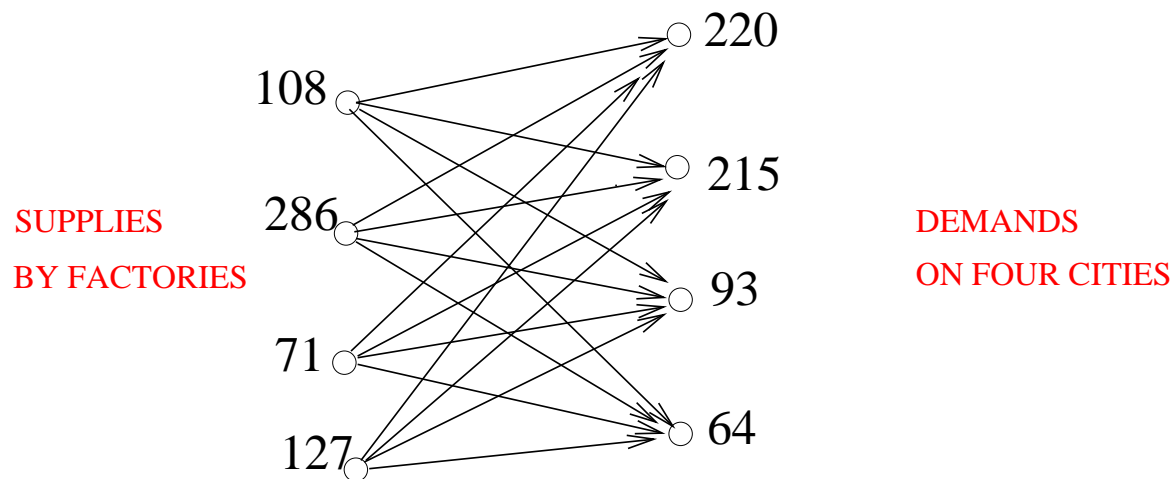


Transportation Polytopes: a Twenty-year Update

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Based on papers with E. Kim, F. Liu (UC Davis),
F. Santos (Cantabria), S. Onn (Technion Haifa), and R. Yoshida (U. Kentucky)



CLASSICAL $(n \times m)$ -transportation Polytopes:

?	?	?	?	220
?	?	?	?	215
?	?	?	?	93
?	?	?	?	64
108	286	71	127	

The 2-way **transportation polytope** is the set of all possible tables whose row/column sums equal the given margins.

Multi-Way Transportation Polytopes.

A d -table of size (n_1, \dots, n_d) is an $n_1 \times n_2 \times \dots \times n_d$ array of nonnegative real numbers $v = (v_{i_1, \dots, i_d})$, $1 \leq i_j \leq n_j$.

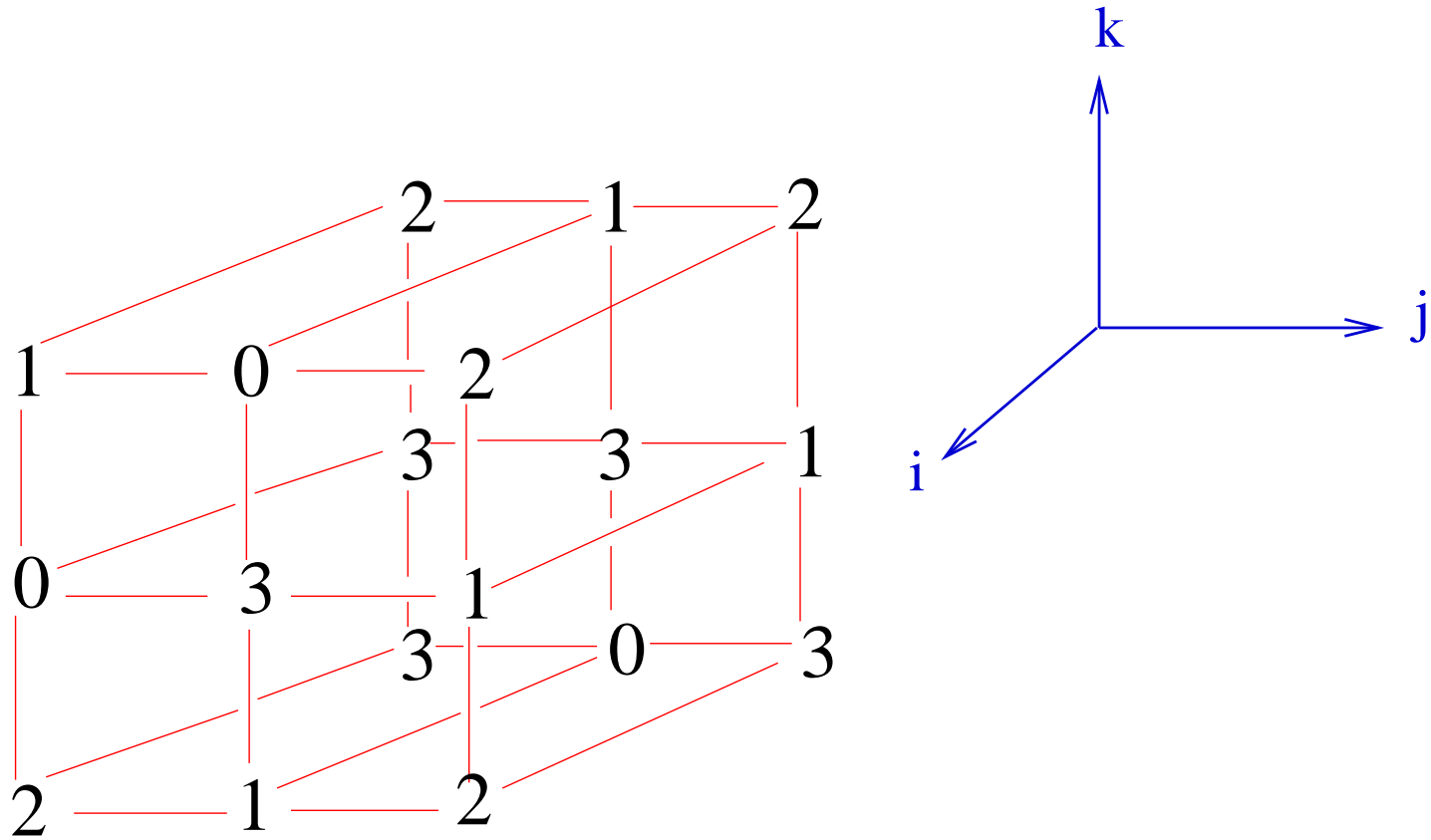
For $0 \leq m < d$, an m -margin of v is any of the possible m -tables obtained by summing the entries over all but m indices.

Example If $(v_{i,j,k})$ is a 3-table then its 0-margin is $v_{+,+,+} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} v_{i,j,k}$, its 1-margins are $(v_{i,+,+}) = (\sum_{j=1}^{n_2} \sum_{k=1}^{n_3} v_{i,j,k})$ and likewise $(v_{+,j,+})$, $(v_{+,+,k})$, and its 2-margins are $(v_{i,j,+}) = (\sum_{k=1}^{n_3} v_{i,j,k})$ and likewise $(v_{i,+,k})$, $(v_{+,j,k})$.

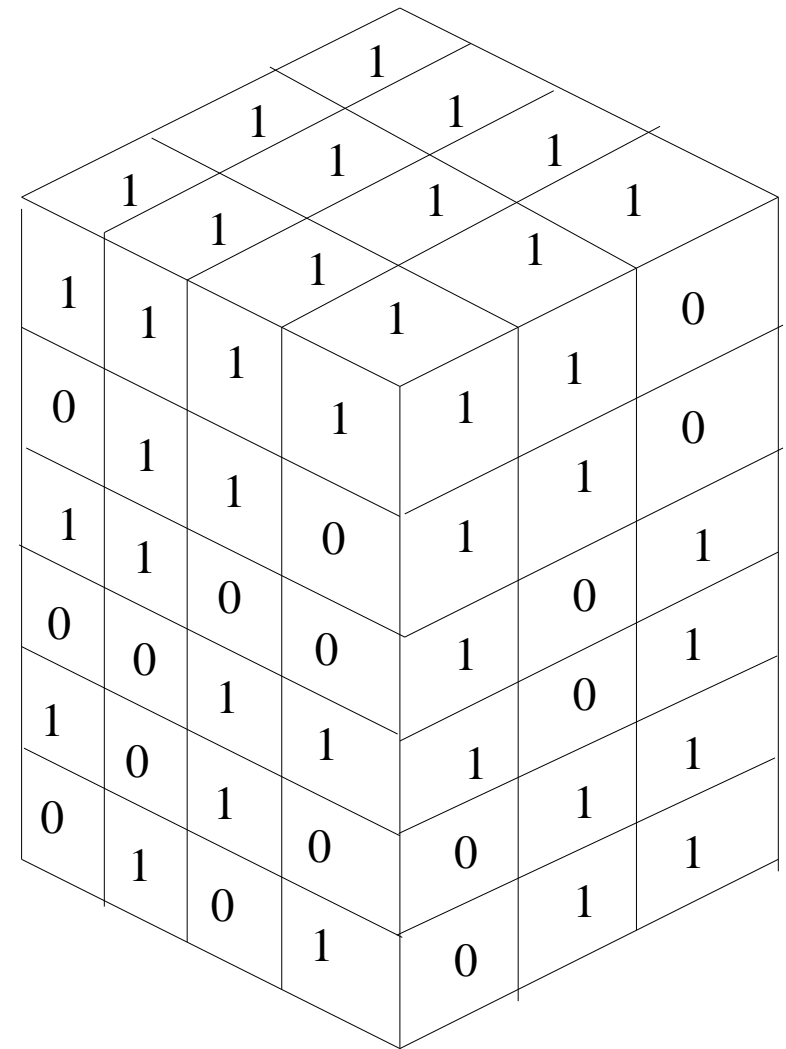
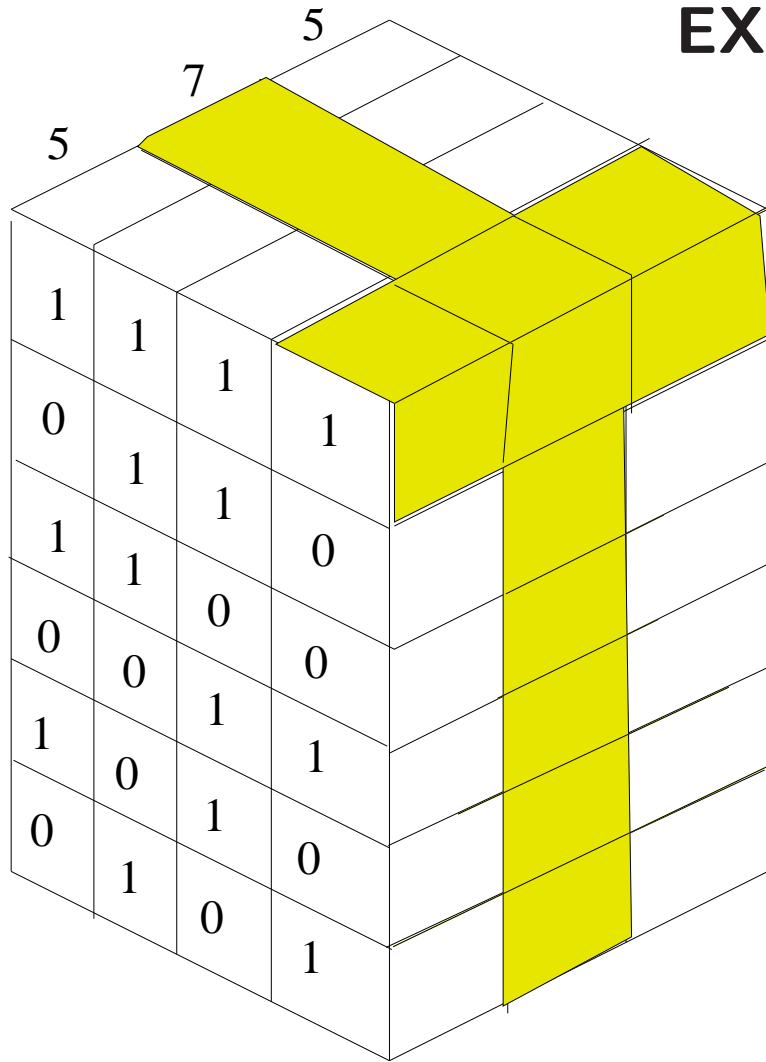
Definition: A **multi-index transportation polytope** is the set of all real d -tables that satisfy a set of given margins.

Definition An **Assignment polytope** is when all right hand-side margins have entry value one. (They also receive the names **Birkhoff-von Neumann polytopes** or **Polytope of poly-stochastic matrices**.)

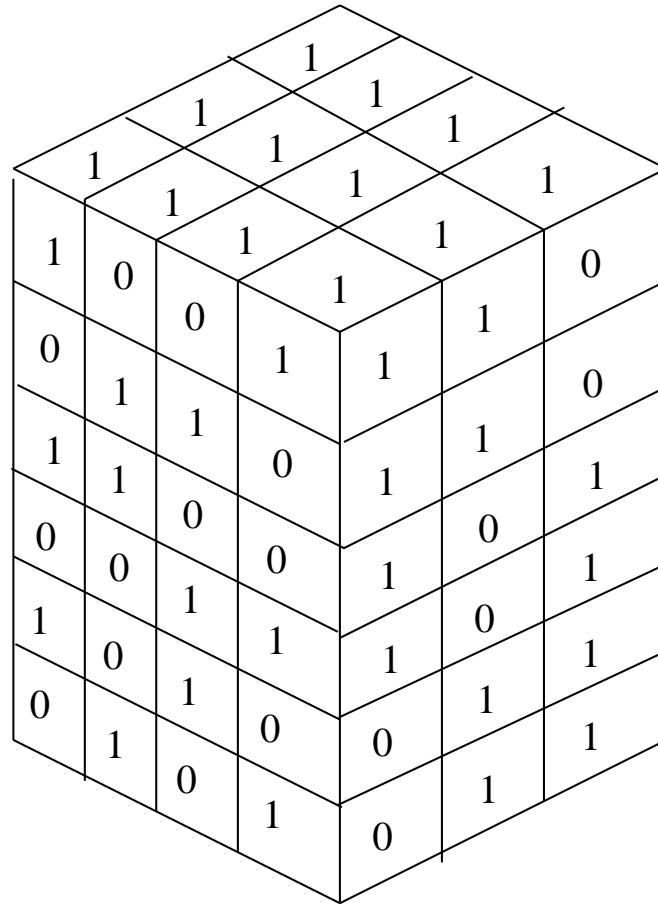
EXAMPLE 1



EXAMPLE 2



EXAMPLE 3



Goals and Plan of this Lecture:

Transportation polytopes arise naturally in Linear and Integer Optimization and Statistics!

There were prior survey monographs and open problem collections proposed in 1960's by Klee-Witzgall and the 1980's by Kovalev-Yemelichev-Kratsov and Vlach:

We present an updated survey on the open problems about these polyhedra.

Part I: Classical (2-way) Transportation Polytopes

Part II: Multiway Transportation Polytopes

We will make clear the differences between the two cases!! If we don't cover your favorite result, sorry! Time is short, see paper version instead!

Part 1:

TWO-WAY TRANSPORTATION POLYTOPES

Constraint Matrix has SPECIAL shape

We can easily see that the system $Ax = b, x \geq 0$ has as constraint matrix A the vertex-edge incidence matrix of the complete bipartite graph $K_{n,m}$.

Example: For the transportation problem

?	?	?	?	220
?	?	?	?	215
?	?	?	?	93
?	?	?	?	64
108	286	71	127	

Jesús De Loera

we have

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Background facts

Theorem: In the classical $n \times m$ transportation polytope we are given 1-marginals $a = (a_1, \dots, a_{n_1})$ and $b = (b_1, \dots, b_m)$,

(1) (dimension and feasibility) If $\sum a_i = \sum b_i$, then polytope is non-empty and it has dimension $(n_1 - 1)(n_2 - 1)$.

(2) (integrality of vertices) Given integral consistent marginals a, b , all vertices of the corresponding transportation polytope are integral. Constraint matrix is **totally unimodular!**

(3) **Theorem:** Let $X = (x_{ij})$ be a matrix that belongs to the $n \times m$ transportation polytope $T(a, b)$. Create the auxiliary graph F_X with $n \times m$ nodes and edge $\{(i, j) : x_{i,j} > 0\}$. Then the matrix X is a vertex of the transportation polytope $T(a, b)$ if and only if F_X forms a forest in the complete bipartite graph $K_{n,m}$.

Vertices of Transportation Polytopes

There is a characterization of which transportation polytopes reach the **largest/smallest** possible number of vertices given by Yemelichev, Kravtsov and collaborators in the 1970's.

Theorem:(Bolker 1972) The maximum possible number of vertices on an $m \times n$ transportation polytopes is achieved by the **central transportation polytope** whose marginals are the m -vector $a^* = (n, n, \dots, n)$ and n -vector $b^* = (m, m, \dots, m)$.

Open Question: What are the possible values that the number of vertices of a simple $m \times n$ transportation polytope can have? Are there gaps or do all integer values on the known interval occur?

Distribution of the number of vertices

sizes	Distribution of number of vertices in transportation polytopes
2×3	3 4 5 6
2×4	4 6 8 10 12
2×5	5 8 11 12 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
3×3	9 12 15 18
3×4	16 21 24 26 27 29 31 32 34 36 37 39 40 41 42 44 45 46 48 49 50 52 53 54 56 57 58 60 61 62 63 64 66 67 68 70 71 72 74 75 76 78 80 84 90 96
4×4	108 116 124 128 136 140 144 148 152 156 160 164 168 172 176 180 184 188 192 196 200 204 208 212 216 220 224 228 232 236 240 244 248 252 256 260 264 268 272 276 280 284 288 296 300 304 312 320 340 360

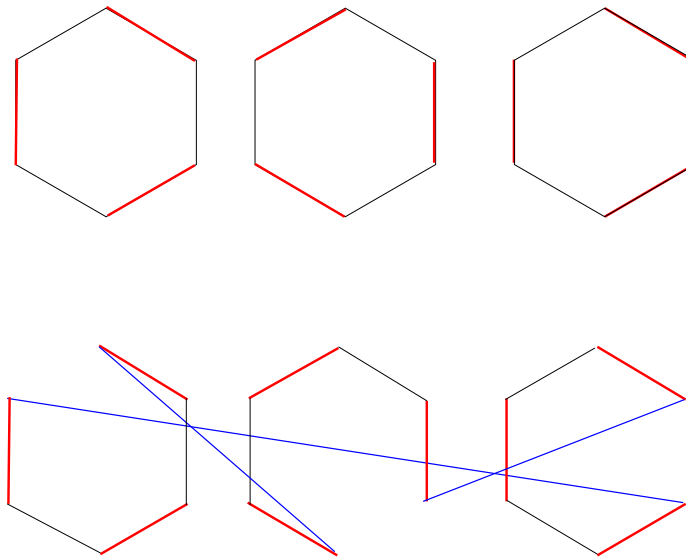
We have attempted a complete classification of **all** combinatorial types of simple $n \times m$ transportation polytopes. 5×5 is the smallest value for which we do not know all types yet!

We use the theory of regular triangulations (Gelfand-Kapranov-Zelevinsky).

Graphs of two-way transportation polytopes

Lemma:(Adjacency conditions of vertices) Let X and Y be distinct vertices of the transportation polytope $T(a, b)$. The vertices are adjacent if and only if the union of the associated graphs F_X and F_Y contains a unique cycle.

Corollary The diameter of the $n \times n$ Birkhoff-von Neumann polytope when $n \geq 4$ is 2.



40-year-old OPEN PROBLEM: What is the largest diameter of the graph of a convex polytope with F facets and dimension d ?

Theorem:(2004 Brightwell-van den Heuvel-Stougie) The diameter of the transportation polytope of size $n \times m$ does not exceed $8(m + n - 2)$.

CONJECTURE: All integer numbers between 1 and $m + n - 1$, and only these, are realized as the diameters of $m \times n$ transportation polytopes.

CONJECTURE: Are all graphs of $n \times m$ transportation polytopes hamiltonian

These conjectures have been proved true for $\min(n, m) \leq 4$.

Lattice points and Volume:

Proposition 2-way transportation polytopes have the “integral interval property” for all entries: If $m < x_{ij} < M$, then there is a lattice point with $x_{ij} = s$ for all integers in between m, M . No gaps!

Proposition It is $\#P$ -hard to count the number of lattice points of a $2 \times n$ transportation polytope. It is hard to count lattice points!

OPEN PROBLEM: Can one find an efficient algorithm to compute the volume of transportation polytopes? For general polytopes this is a $\#P$ -hard problem!

OPEN PROBLEM: Determine the exact value for the **volume** of the $n \times n$ Birkhoff-von Neumann polytope.

Can this volume be computed efficiently? Can you find a nice formula?
(R. Stanley).

Theorem: (JD, F. Liu, R. Yoshida) For any choice of fixed $\ell \in [n]$, the normalized volume of the polytope B_n of doubly stochastic $n \times n$ matrices is given

$$\text{vol}(B_n) = \frac{1}{((n-1)^2)!} \sum_{\sigma \in S_n} \sum_{T \text{ arborescence of } K_n} \frac{\langle c, \sigma \rangle^{(n-1)^2}}{\prod_{e \notin E(T)} \langle c, W^{T,e} \sigma \rangle}. \quad (1)$$

$W^{T,e}(i, j)$ denotes an $n \times n$ $(0, -1, 1)$ -matrix associated to the unique oriented cycle in $T + e$. $W^{T,e} \sigma$ denotes the usual matrix multiplication of $W^{T,e}$ and the permutation matrix σ . Finally, c is any real vector such that $\langle c, W^{T,e} \sigma \rangle$ is non-zero for all pairs (T, e) .

Part 2:

MULTI-WAY TRANSPORTATION POLYTOPES

Several Open Questions from the 1980's

(0) **Real feasibility and Dimension:** Is there a characterization in terms of the 2-margins of those 3-way transportation polytopes which are empty? What is the dimension of the polytope?

Several conditions on the margins proposed by Schell, Haley, Moravek and Vlack during the 1980's. Do they suffice to guarantee that the polytope is non-empty?

What are the possible dimensions of an $n \times m \times k$ transportation polytopes? Is it always equal to $(n - 1)(m - 1)(k - 1)$?

(1) **Integer Feasibility Problem:** Given a prescribed collection of integral margins that seem to describe a d -transportation polytope of size (n_1, \dots, n_d) ,

Does there really exist an integer table with these margins? can one such integral d -table be efficiently determined ?

(2) **Table Entry Problems:** Given a collection of margins coming from an actual (but unknown) d -table, and an index tuple (i_1, \dots, i_d) , Decide whether a single value (hence that in the actual table) occurs as the entry v_{i_1, \dots, i_d} in any table v having the given margins ?

Do k -way tables have the “interval property” for entry values?

(3) **The graphs of k -way transportation polytopes**

Can one estimate minimum/maximum number of vertices possible? Do they have a nice easy characterization?

Do we have a good bound for the diameter? Is the Hirsch conjecture true in this case?

ANSWERS!

(0) (**Vlach Problems 1986**): Do the conditions on the margins proposed by Schell, Haley, Moravek and Vlach suffice to guarantee that the polytope is non-empty?

NO!

What are possible dimensions?

Dimension can be any number from 1 up to $(n - 1)(m - 1)(k - 1)$

Is there an efficient characterization in terms of the 2-marginals of those 3-way transportation polytope which are empty? or the dimension of the polytope?

Unlikely, unless $NP=P$

(1) **Table Feasibility Problem:** Given a prescribed collection of marginals that seem to describe a d -transportation polytope of size (n_1, \dots, n_d) ,

Does there really exist an integer table with these marginals? can one such integral d -table be efficiently determined ?

NO, problem is NP-hard for $3 \times n \times m$ transportation polytopes (improves on results of Irving and Jerrum).

(2) **Table Entry-Security Problem:** Given a collection of margins coming from an actual (but unknown) d -table, and an index tuple (i_1, \dots, i_d) , Decide whether a single value (hence that in the actual table) occurs as the entry v_{i_1, \dots, i_d} in any table v having the given margins ?

NO, problem is NP-hard for $3 \times n \times m$ transportation polytopes(improves on results of Irving and Jerrum).

Any set of non-negative integers can be the values for a fixed entry! Gaps!

Theorem: (JD & S. Onn 2004) Any rational polytope $P = \{y \in \mathbb{R}^n : Ay = b, y \geq 0\}$ is polynomial-time representable as a 3-way $(r \times c \times 3)$ transportation polytope

$$T = \{x \in \mathbb{R}_{\geq 0}^{r \times c \times 3} : \sum_i x_{i,j,k} = w_{j,k}, \sum_j x_{i,j,k} = v_{i,k}, \sum_k x_{i,j,k} = u_{i,j}\}.$$

The polytope $P \subset \mathbb{R}^p$ is *representable* as a polytope $T \subset \mathbb{R}^q$ if there is an injection $\sigma : \{1, \dots, p\} \rightarrow \{1, \dots, q\}$ such that the projection

$$\pi : \mathbb{R}^q \rightarrow \mathbb{R}^p : x = (x_1, \dots, x_q) \mapsto \pi(x) = (x_{\sigma(1)}, \dots, x_{\sigma(p)})$$

is a bijection between T and P and between the sets of integer points $T \cap \mathbb{Z}^q$ and $P \cap \mathbb{Z}^p$.

Theorem (JD & S. Onn)(now with complexity estimate).

Any polytope $P = \{y \in \mathbb{R}_{\geq 0}^n : Ay = b\}$ with integer $m \times n$ matrix $A = (a_{i,j})$ and integer b is polynomial-time representable as a slim transportation polytope

$$T = \left\{ x \in \mathbb{R}_{\geq 0}^{r \times c \times 3} : \sum_i x_{i,j,k} = w_{j,k}, \sum_j x_{i,j,k} = v_{i,k}, \sum_k x_{i,j,k} = u_{i,j} \right\},$$

with $r = O(m^2(n + L)^2)$ rows and $c = O(m(n + L))$ columns,

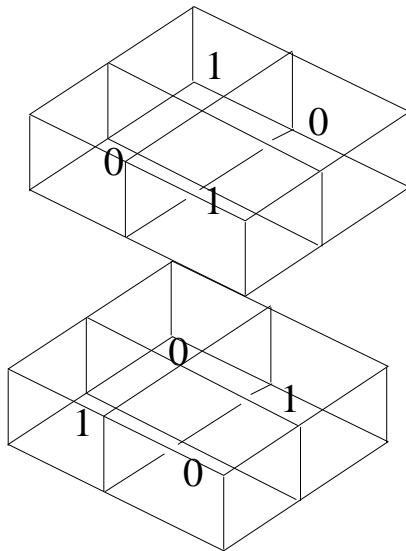
where $L := \sum_{j=1}^n \max_{i=1}^m \lceil \log_2 |a_{i,j}| \rceil$.

EXAMPLE: $P = \{y \mid \underset{\text{INPUT}}{2y} = \underset{\text{OUTPUT}}{1}, y \geq 0\}$

table size (2,2,2)

1-marginals = 1

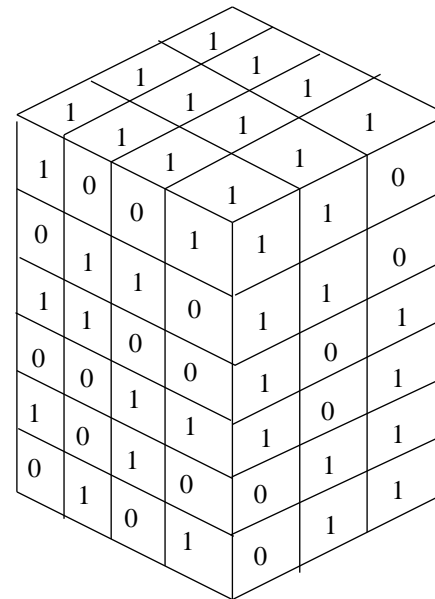
entry upper bounds (see below)



Unique real feasible array
 All entries = 1/2 or 0
 No integer table

table size (3,4,6)

2-marginals (see below)



Unique real feasible array
 All entries 1/2 and 0
 No integer table.

MORE CONSEQUENCES

(3) **All linear and integer programming problems are Slim 3-way Transportation Problems.** Any linear or integer programming problem is equivalent to one that has only $\{0, 1\}$ -valued coefficients, with exactly three 1's per column, and depends only on the right-hand side data.

(4) **Problem reduction** If there is an strongly-polynomial-time algorithm for 3-way transportation problems would imply the existence of a strongly-polynomial-time algorithm for linear programs in general. Similarly, if the Hirsch conjecture is true for 3-way transportation polytopes given by 2-marginals, then it is true for all rational polytopes!

(5) **Hardness of approximation** The family of 3-way transportation problems of $(r, c, 3)$ and specified by 2-margins contains subproblems that admit fully polynomial approximation schemes as well as subproblems that do not have arbitrarily close approximation (unless $NP = P$).

k -way 1-margin transportation problems

Theorem: (JD & S. Onn 2004) Any rational polytope $P = \{y \in \mathbb{R}^n : Ay = b, y \geq 0\}$ is polynomial-time representable as a **face** of a 3-way $(r \times c \times 3)$ transportation polytope with 1-margins

$$T = \{x \in \mathbb{R}_{\geq 0}^{r \times c \times 3} : \sum_{i,j} x_{i,j,k} = w_k, \sum_{j,k} x_{i,j,k} = v_i, \sum_{i,k} x_{i,j,k} = u_j\}.$$

Theorem: [JD, E. Kim, F. Santos, S. Onn 2007] Any 3-way transportation polytope of size $l \times m \times n$ with fixed 1-margins has diameter no more than $8(l + m + n)^2$.

Question: How much bigger can the diameter of a face be??

k -way 1-margin Assignment polytopes are Nasty too!!

Theorem: (Karp 1972) The optimization problems

maximize/minimize $\sum p_{i,j,k} x_{i,j,k}$ subject to

$$\{ x \in \mathbb{Z}_{\geq 0}^{n \times n \times n} : \sum_{i,j} x_{i,j,k} = 1, \sum_{j,k} x_{i,j,k} = 1, \sum_{i,k} x_{i,j,k} = 1 \} .$$

are NP-hard.

Theorem: (Crama-Spieksma 1992) For the minimization case, no polynomial time algorithm can even achieve a constant performance ratio unless NP=P.

Theorem (Gromova 1992) Any positive decreasing vector v with components less than one is a permutation of part of a vertex of some 3-way assignment polytope.

MORE OPEN PROBLEMS/ QUESTIONS

- **Open Question:** What is the distribution of number of vertices for $m \times n \times k$ non-degenerate 3-way transportation polytopes, specified by 2-margin matrices A, B, C ?
- **Open Question:** What is the largest possible number of vertices in a 3-way $n \times m \times k$ transportation polytope?

The YKK conjecture The $m \times n \times k$ *generalized central transportation polytope* is the 3-way transportation polytope of line sums whose 2-margins are given by the $n \times k$ matrix $U(j, k) = m$, the $m \times k$ matrix $V(i, k) = n$, and the $m \times n$ matrix $W(i, j) = k$.

Conjecture: (Yemelichev-Kovalev-Kratsov) The generalized central transportation polytope has the largest number of vertices among 3-way transportation polytopes.

FALSE!! (JD, E. Kim, F. Santos, S. Onn 2007)

Here are explicit 2-marginals for a $3 \times 3 \times 3$ transportation polytope which have more vertices (270 vertices) than the generalized central transportation polytope, with only 66 vertices:

164424	324745	127239	163445	49395	403568	184032	123585	2692
262784	601074	9369116	1151824	767866	8313284	886393	6722333	9355
149654	7618489	1736281	1609500	6331023	1563901	1854344	302366	90759