

Gelfand–Tsetlin polytopes and toric degenerations of the invariants of n points in \mathbb{P}^2

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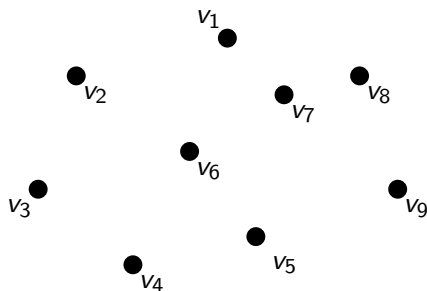
Outline

- 1 Setup — Invariants of n points in \mathbb{P}^2
- 2 Degeneration to toric ring — Motivation from \mathbb{P}^1 case
- 3 Main result — toric degeneration generated in exponential degree
- 4 What went wrong? — An essential generator with exponential degree
- 5 Conclusion

n labeled points in \mathbb{P}^2

Let v_1, \dots, v_n be labeled points in \mathbb{P}^2 . Assume general position — no three on a line. Assume also 3 divides n .

This is a point in $v \in (\mathbb{P}^2)^n$.

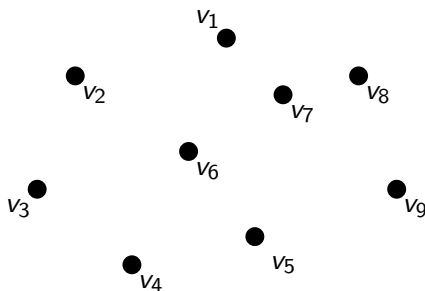


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$SL_3(\mathbb{C})$ acts on v . We study invariants of the orbits $SL_3(\mathbb{C})v$.

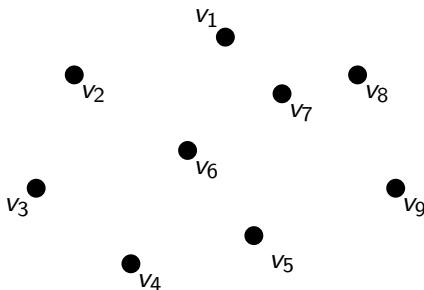


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Definition

Let R be the ring of projective invariants for this action.

Constructing ring of invariants R using tableaux

- Let $\lambda = \underbrace{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}} \cdots \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$ and $\mu = \underbrace{(1, \dots, 1)}_{n \text{ terms}}$.

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 $n/3$ columns
- For tableau τ with shape $N\lambda$ and content $N\mu$, $N \geq 0$, let entries of j th column be a_{1j}, a_{2j}, a_{3j} .

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- For tableau τ with shape $N\lambda$ and content $N\mu$, $N \geq 0$, let entries of j th column be a_{1j}, a_{2j}, a_{3j} . Put

$$f_{\tau} = \prod_{j=1}^{n/3} \det_{a_{1j}, a_{2j}, a_{3j}}$$

where $\det_{a_{1j}, a_{2j}, a_{3j}} : SL_n(\mathbb{C}) \rightarrow \mathbb{C}$ is determinant of submatrix with rows 1, 2, 3 and columns a_{1j}, a_{2j}, a_{3j} .

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- Let $R_N = \text{Span}\{f_{\tau} : \tau \text{ has shape } N\lambda \text{ and content } N\mu\}$.

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Fact

Ring of invariants R is the graded ring

$$R = \bigoplus_{N \geq 0} R_N.$$

The Question

R is finitely generated as an algebra.

Question

What is minimum d such that R is generated by $\bigoplus_{N=0}^d R_N$?

Degeneration to toric ring — Motivation from \mathbb{P}^1 case

Theorem (Kempe 1894)

For case of points on line \mathbb{P}^1 , R is generated in degree 2.

Theorem (Howard, Millson, Snowden, Vakil 2006)

Relations on these degree-2 generators are at most quartic.

HMSV proved this using a degeneration of R to semi-group ring over a **Gelfand–Tsetlin** polytope.

The degeneration to toric ring

Using tools of Gonciulea–Lakshmibai (1996) and Foth–Hu (2005), we can degenerate R to a graded toric ring R' such that

- The positives:
 - ▶ Relations in R' are simpler: ideal is binomial.
 - ▶ Generating set in R' lifts to generating set of R with same degrees.
- The negative: A minimal generating set in R' may lift to a **non-minimal** generating set in R .

A promising prelude: In \mathbb{P}^1 case, R' is generated in degree 2 for all n . Hence so is R .

Main result — toric degeneration generated in exponential degree

Theorem (Howard and M.)

Let R be ring of invariants for n points in \mathbb{P}^2 . Then R is generated in degree at worst $O(n^2)$.

However, the toric degeneration R' requires generators of degree exponential in n .

Bound on generators of R comes from general results of Harm Derksen on invariant polynomials (2001).

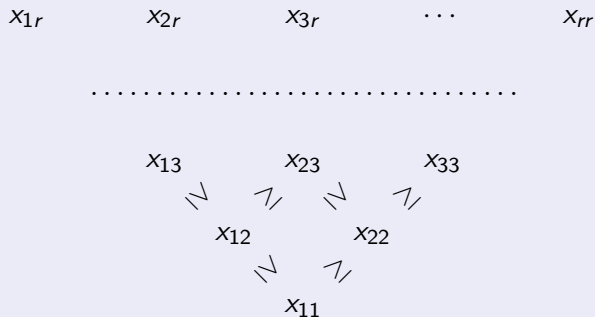
Question

What goes wrong in the toric degeneration R' ?

Constructing the toric degeneration — Definition of GT-patterns

Definition

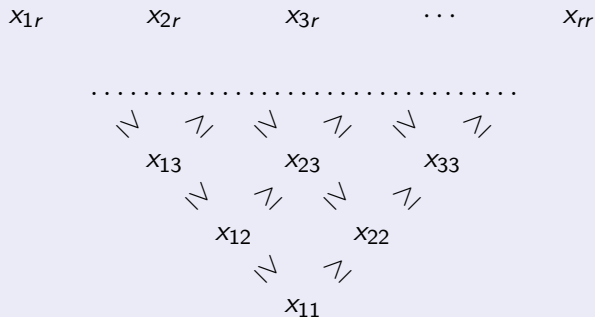
A **Gelfand–Tsetlin (GT) pattern** is a triangular array of real numbers satisfying interlacing inequalities



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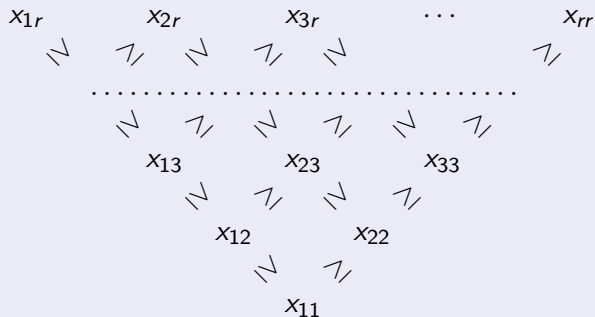
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Definition of GT-polytopes

Definition (Gelfand, Tsetlin 1950)

Given $\tau, \beta \in \mathbb{Z}^n$, the **GT-polytope** $GT(\tau, \beta)$ is the set of GT-patterns $(x_{ij})_{1 \leq i \leq j \leq n}$ such that

- top row is τ
- row sum of j th row from bottom is $\sum_{i=1}^j \beta_j$.

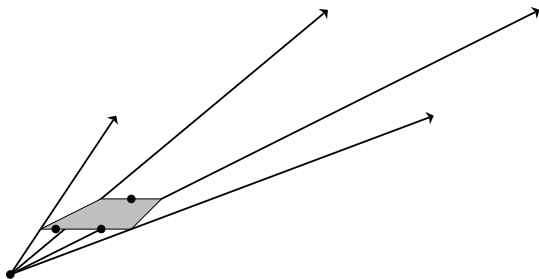
Example

$$\begin{array}{cccc} 5 & 5 & 3 & 1 \\ & 5 & 4 & 1 \\ & & 5 & 2 \\ & & & 2 \end{array} \in GT(5531, 2534)$$

The semigroup ring generated by a polytope

A rational polytope $P \subset \mathbb{R}^D$ sitting in $x_D = 1$ hyperplane generates a semigroup

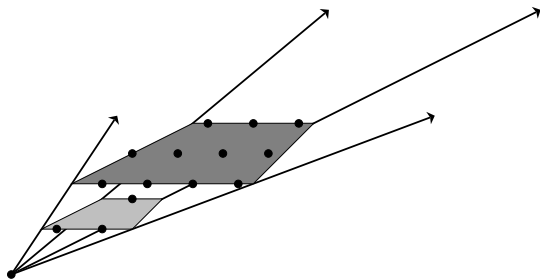
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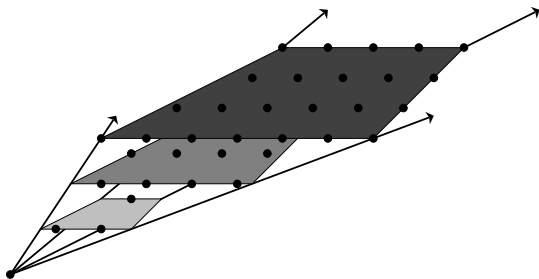
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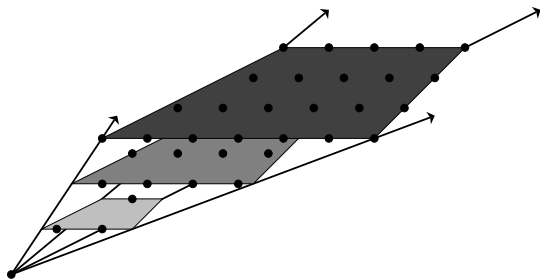
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Definition

The **semigroup ring** over a polytope P is $\mathbb{C}[S_P]$.

Exponential growth of degrees of generators

Let $k = n/3$. Recall that $\lambda = \underbrace{(k, k, k, 0, \dots, 0)}_{n \text{ terms}}$ and $\mu = \underbrace{(1, \dots, 1)}_{n \text{ terms}}$. Let

$P = GT(\lambda, \mu)$. Then $\mathbb{C}[S_P]$ is a toric degeneration of R .

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Theorem (De Loera and M. (2004))

GT-polytopes may have nonintegral vertices.

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Theorem (De Loera and M. (2004))

GT-polytopes may have nonintegral vertices.

Theorem (Howard and M.)

Let $P = GT(\lambda, \mu)$. Then P has a vertex with exponential denominator. Hence, $\mathbb{C}[S_P]$ has essential generators of exponential degree.

Constructing the vertex — Recurrence relations for entries

Definition

Put $k = n/3$. Define coupled recurrence relations

$$T_0^{(1)} = k$$

$$T_0^{(2)} = k - 1/2$$

$$T_j^{(1)} = T_{j-1}^{(2)} - 1 \quad (j \geq 1)$$

$$T_j^{(2)} = \frac{1}{2} \left(T_j^{(1)} + T_{j-1}^{(1)} \right) \quad (j \geq 1).$$

k k k
 k k
 k
 k

Constructing the vertex
— Arranging entries in
pattern

$$\begin{array}{ccccc}
 k & & k & & k \\
 & k & & k & - 1 \\
 & & k & & k - \frac{1}{2} \\
 & & & k & \\
 & & & & k
 \end{array}$$

Constructing the vertex
 — Arranging entries in
 pattern

$$\begin{array}{ccccc}
 k & & k & & k \\
 & k & & k & - 1 \\
 & & k & & k - \frac{1}{2} & T_1^{(1)} \\
 & & & k & & T_1^{(1)} & T_1^{(1)} \\
 & & & & & & T_1^{(1)} \\
 & & & & & & & T_1^{(1)}
 \end{array}$$

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$$\begin{array}{ccccc}
 k & & k & & k \\
 & k & & k & - 1 \\
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& & & & & & & T_2^{(1)} \\
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\end{array}$$

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$$\begin{array}{cccc}
k & & k & & k \\
& k & & k & - 1 \\
& & k & - \frac{1}{2} & T_1^{(1)} \\
& & & k & T_1^{(1)} & T_1^{(1)} \\
& & & & T_1^{(2)} & T_1^{(1)} & T_2^{(1)} \\
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$$\begin{array}{cccc}
k & & & \\
& k & & \\
& & k & k-1 \\
& & & k - \frac{1}{2} & T_1^{(1)} \\
& & k & & T_1^{(1)} & T_1^{(1)} \\
& & & T_1^{(2)} & T_1^{(1)} & T_2^{(1)} \\
& & & & T_1^{(1)} & T_2^{(1)} & T_2^{(1)} \\
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& & & & & & T_2^{(1)} & \ddots
\end{array}$$

Constructing the vertex
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$$\begin{array}{ccccccc}
k & & k & & k & & \\
& k & & k & & k-1 & \\
& & k & & k-\frac{1}{2} & & T_1^{(1)} \\
& & & k & & T_1^{(1)} & T_1^{(1)} \\
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& & & & & & T_2^{(2)} & & T_2^{(1)} \\
& & & & & & & T_2^{(1)} & \cdots & T_{N-1}^{(1)} \\
& & & & & & & & & T_{N-1}^{(1)} & T_{N-1}^{(1)} \\
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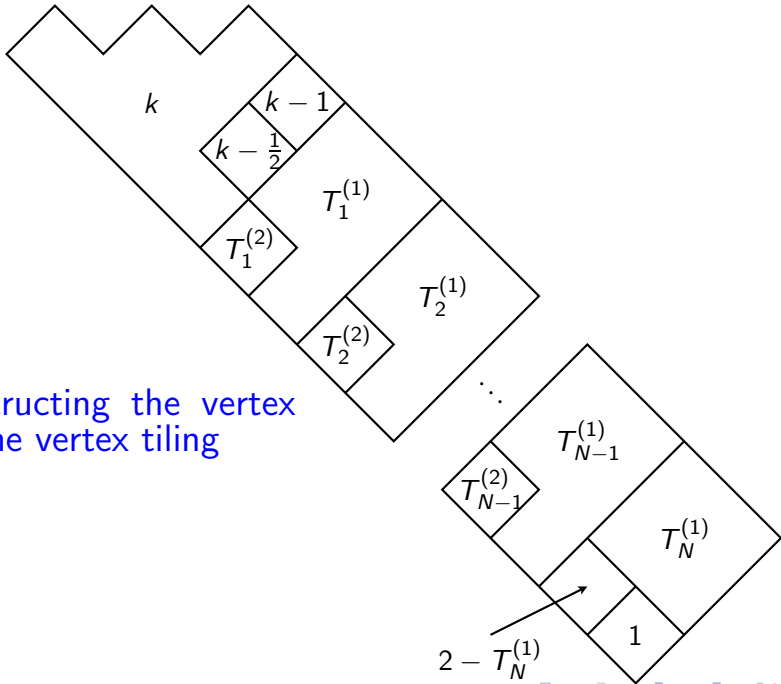
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k & & k & & k & & \\
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& & & k & & T_1^{(1)} & T_1^{(1)} \\
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& & & & & T_1^{(1)} & & T_2^{(1)} & T_2^{(1)} \\
& & & & & & T_2^{(2)} & & T_2^{(1)} \\
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& & & & & & & & & T_{N-1}^{(1)} & T_{N-1}^{(1)} \\
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$$\begin{array}{ccccccc}
k & & k & & k & & \\
& k & & k & & k-1 & \\
& & k & & k-\frac{1}{2} & & T_1^{(1)} \\
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& & & & & & T_2^{(2)} & & T_2^{(1)} \\
& & & & & & & T_2^{(1)} & & \ddots & T_{N-1}^{(1)} \\
& & & & & & & & T_{N-1}^{(1)} & & T_{N-1}^{(1)} \\
& & & & & & & & & T_{N-1}^{(2)} & T_{N-1}^{(1)} & T_N^{(1)} \\
& & & & & & & & & & T_{N-1}^{(1)} & T_N^{(1)} & T_N^{(1)} \\
& & & & & & & & & & & & T_N^{(1)} \\
& & & & & & & & & & & & & T_N^{(1)} \\
& & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & & & 2 - T_N^{(1)}
\end{array}$$

Constructing the vertex
— Arranging entries in
pattern



Constructing the vertex
— The vertex tiling

Conclusion

- The toric degeneration of the ring of invariants for n points in \mathbb{P}^2 requires generators of exponential degree.
- However, the ring of invariants itself is at worst generated in polynomial degree.
- Hence, the toric degeneration so effective in \mathbb{P}^1 case fails in \mathbb{P}^2 case.